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A NUMERICAL STUDY OF VISCOUS BREAKING WATER WAVES IN THE HIGH REYNOLDS NUMBER LIMIT

GDR MATHGEOPHY - CONFÉRENCE JEUNES

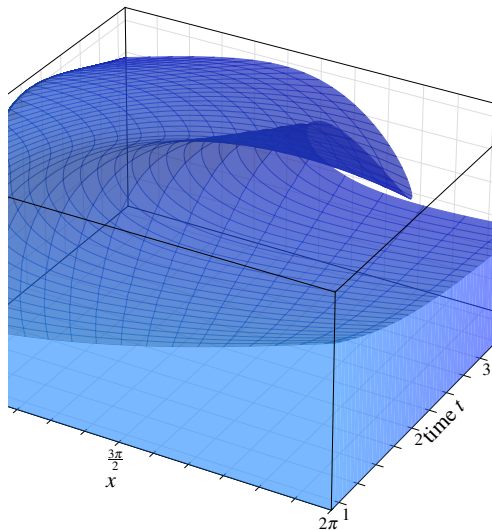
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January 25, 2024

Viscous Water Waves

1. Introduction: the Water Waves problem
2. Introducing the viscosity
3. Numerical methods
4. The $\text{Re} \rightarrow +\infty$ limit
5. Surface Boundary Layer
6. Effects of viscosity on free surfaces



The Water Waves Problem

Neglecting viscosity...
Neglecting vorticity...



The Water Waves Problem

Euler equation

Incompressible Euler equation in Ω_t :

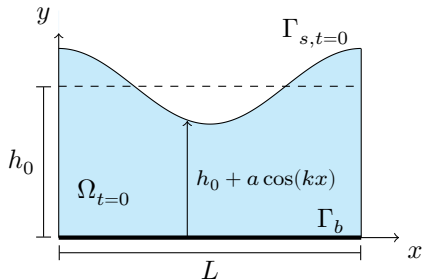
$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\nabla p + \mathbf{g} \\ \nabla \cdot \mathbf{u} &= 0 \end{cases}$$

Boundary condition on Γ_b :

$$\mathbf{u} \cdot \mathbf{n} = 0$$

Boundary condition on $\Gamma_{s,t}$:

$$p = 0$$

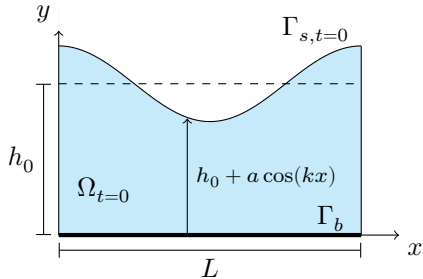


The Water Waves Problem

Domain advection

Surface parametrized by $\gamma(t, s)$. Advection with the fluid,

$$\partial_t \gamma(t, s) = \mathbf{u}(\gamma(t, s))$$



The Water Waves Problem

Velocity Potential

We **suppose** that $\nabla \times \mathbf{u} = 0$, so that

$$\mathbf{u} = \nabla \phi,$$

solution of the Laplace equation in Ω_t ,

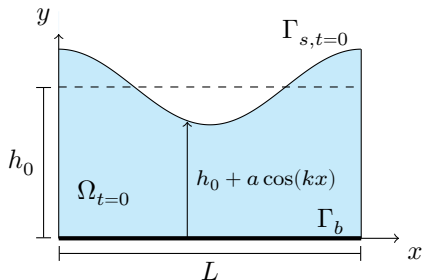
$$\Delta \phi = 0.$$

Boundary condition on Γ_b :

$$\partial_n \phi = 0.$$

Boundary condition on $\Gamma_{s,t}$ (**Bernoulli's** equation),

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + gy = 0.$$



The Water Waves Problem

Dirichlet-Neumann operator

If $\Gamma_{s,t}$ is a graph ($\gamma(t, s) \rightarrow \gamma(t, x)$), the interface advection equation becomes

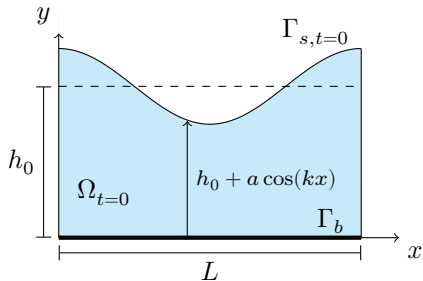
$$\partial_t \gamma = \sqrt{1 + (\partial_x \gamma)^2} \partial_n \phi|_{y=\gamma}$$

We define the **trace** ψ of the potential ϕ ,

$$\psi(t, x) = \phi(t, x, \gamma(t, x)),$$

and the **Dirichlet-Neumann** operator,

$$(G[\gamma]\psi)(t, x) = \sqrt{1 + (\partial_x \gamma)^2} \partial_n \phi|_{y=\gamma} = \partial_t \gamma$$



The Water Waves Problem

Common form

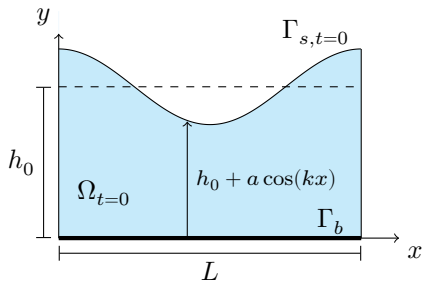
$$\partial_t \gamma = G[\gamma] \psi$$

$$\partial_t \psi + g\gamma + \frac{1}{2}(\partial_x \psi)^2 - \frac{1}{2} \frac{(G[\gamma] \psi + (\partial_x \gamma)(\partial_x \psi))^2}{1 + (\partial_x \gamma)^2} = 0$$

+ initial conditions.

Non-local, non-linear equations with Hamiltonian structure for functions (γ, ψ) defined **only** on the **interface**!

Can also be done for parametrized interfaces $\gamma(t, s)$, allowing overturning waves (Craig, 2017).



Viscous Water Waves

Adding viscosity!

Adding vorticity!



Viscous Water Waves

Nondimensionalization

Nondimensional quantities are defined as follows

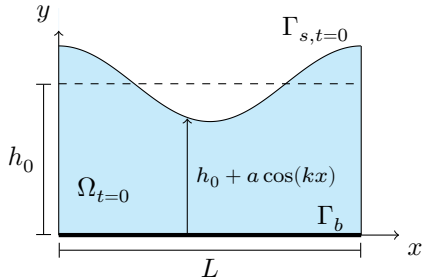
$$x \rightarrow h_0 x$$

$$u \rightarrow \sqrt{gh_0} \cdot u$$

$$p \rightarrow \rho gh_0 \cdot p$$

This allows to define the Reynolds number Re ,

$$\text{Re} = \frac{\rho h_0 \sqrt{gh_0}}{\mu}$$



Viscous Water Waves

Navier-Stokes equation

Incompressible, non-dimensional, Navier-Stokes equation in Ω_t :

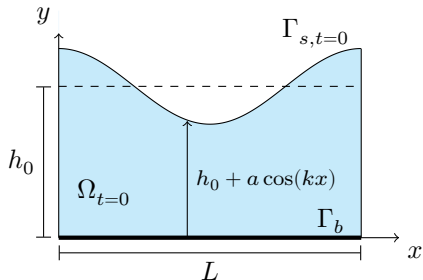
$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\nabla p + \frac{1}{\text{Re}} \Delta \mathbf{u} + \mathbf{g} \\ \nabla \cdot \mathbf{u} &= 0 \end{cases}$$

Navier boundary conditions on Γ_b ,

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad ; \quad \mathbf{t} \cdot [\nabla \mathbf{u} + (\nabla \mathbf{u})^t] \cdot \mathbf{n} = 0$$

Stress-free boundary conditions on $\Gamma_{s,t}$,

$$p\mathbf{n} - \frac{1}{\text{Re}} \cdot [\nabla \mathbf{u} + (\nabla \mathbf{u})^t] \cdot \mathbf{n} = 0$$



Initial conditions

Linear wave

We know many **irrotational** solutions of the **non-viscous** water wave problem over a **flat bottom** (e.g. Stokes waves, Gertsner waves, solitary waves).

Linear waves of (small) amplitude a ,

$$\gamma(t, x) = a \cos(kx - \omega t) \quad \text{with} \quad \omega = \sqrt{gk \tanh(kh_0)}$$
$$\phi(t, x, y) = \frac{a\omega}{k} \frac{\cosh(ky)}{\sinh(kh_0)} \cdot \sin(kx - \omega t) + \mathcal{O}(ka)$$

The velocity is then

$$\mathbf{u}(t, x, y) = \nabla \phi = \frac{a\omega}{\sinh(kh_0)} \cdot \begin{bmatrix} \cosh(ky) \cos(kx - \omega t) \\ \sinh(ky) \sin(kx - \omega t) \end{bmatrix}$$

Initial conditions

Laplace problem

Initial cosine interface $\gamma(0, x) = a \cos(kx)$.

Initial irrotational velocity $\mathbf{u}_0 = \nabla \phi$ with ϕ solution of the Laplace problem

$$\Delta \phi = 0 \quad \partial_n \phi = 0 \text{ on } \Gamma_b \quad \partial_n \phi = \mathbf{u}_0 \cdot \mathbf{n} \text{ given on } \Gamma_{s,0}$$

We know that

$$\mathbf{n}(x) = \frac{1}{\sqrt{1 + (\partial_x \gamma)^2}} \begin{bmatrix} -\partial_x \gamma \\ 1 \end{bmatrix}$$

and we choose

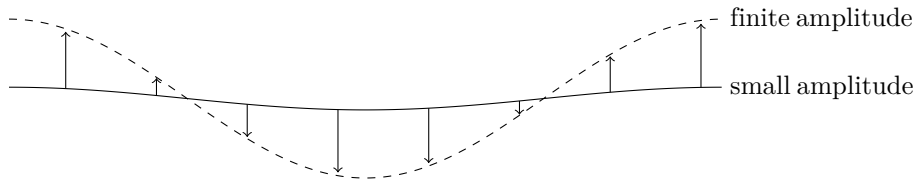
$$\mathbf{u}_0|_{y=\gamma(0,x)} = a\omega \begin{bmatrix} \tanh^{-1}(kh_0) \cos(kx) \\ \sin(kx) \end{bmatrix}$$

i.e. putting $y = h_0$ in the linear wave solution.

Initial conditions

Finite amplitude extension

$$u_0|_{y=\gamma(0,x)} = \frac{a\omega}{\sinh(kh_0)} \cdot \begin{bmatrix} \cosh(ky) \cos(kx) \\ \sinh(ky) \sin(kx) \end{bmatrix} \longrightarrow a\omega \begin{bmatrix} \tanh^{-1}(kh_0) \cos(kx) \\ \sin(kx) \end{bmatrix}$$



u_0 evaluated for $y = h_0$ (small amplitude) and mapped on $y = \gamma(0, x)$ (finite amplitude).

This is a purely arbitrary way of constructing the initial condition.

Weak formulation

Viscous water waves

Function space

$$\mathbf{H}_{\Gamma_b}^1(\Omega_t) = \left\{ \mathbf{v} \in (H^1(\Omega_t))^2 : \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_b \right\}$$

We do **not** suppose $\nabla \cdot \mathbf{u} = 0$ in the function space (would not work with finite elements).

Let $\mathbf{v} : \Omega_t \rightarrow \mathbb{R}^2$ be a test (vector) function. We multiply **Navier-Stokes** by $\mathbf{v}$ and integrate over Ω_t ,

$$\int_{\Omega_t} \mathbf{v} \cdot \partial_t \mathbf{u} + \mathbf{v} \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} - \frac{1}{\text{Re}} \mathbf{v} \cdot \Delta \mathbf{u} + \mathbf{v} \cdot \nabla p - \mathbf{v} \cdot \mathbf{g} = 0$$

Let $q : \Omega_t \rightarrow \mathbb{R}$ a test (scalar) function. We multiply the incompressibility condition by q and integrate over Ω_t ,

$$\int_{\Omega_t} q \nabla \cdot \mathbf{u} = 0$$

Weak formulation

Integration by parts

Green formulas,

$$\begin{aligned}\int_{\Omega_t} (\mathbf{v} \cdot \Delta \mathbf{u} + \nabla \mathbf{v} : \nabla \mathbf{u}) &= \int_{\partial \Omega_t} \mathbf{v} \cdot \nabla \mathbf{u} \cdot \mathbf{n} \, dS \\ \int_{\Omega_t} (\mathbf{v} \cdot \nabla p + p \nabla \cdot \mathbf{v}) &= \int_{\partial \Omega_t} p \mathbf{v} \cdot \mathbf{n} \, dS\end{aligned}$$

This leads to

$$\begin{aligned}\int_{\Omega_t} \left(\mathbf{v} \cdot \nabla p - \frac{1}{\text{Re}} \mathbf{v} \cdot \Delta \mathbf{u} \right) &= \int_{\Omega_t} \left(\frac{1}{\text{Re}} \nabla \mathbf{v} : \nabla \mathbf{u} - p \nabla \cdot \mathbf{v} \right) \\ &\quad + \int_{\partial \Omega_t} \left(p \mathbf{v} \cdot \mathbf{n} - \frac{1}{\text{Re}} \mathbf{v} \cdot \nabla \mathbf{u} \cdot \mathbf{n} \right) dS\end{aligned}$$

Weak formulation

Boundary term (I)

The boundary $\partial\Omega_t$ can be decomposed as such,

$$\partial\Omega_t = \Gamma_b \cup \Gamma_{s,t} \cup \text{left} \cup \text{right}$$

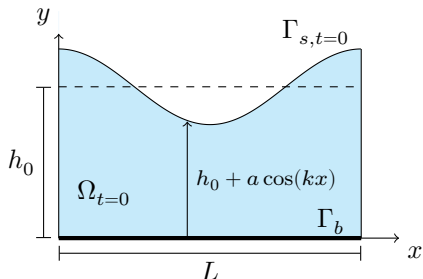
The periodicity implies

$$\int_{\text{left}} (\dots) \cdot \mathbf{n} \, dS = \int_{\text{right}} (\dots) \cdot (-\mathbf{n}) \, dS$$

so each term annihilates the other.

On $\Gamma_{s,t}$ we use the stress-free boundary condition,

$$\int_{\Gamma_{s,t}} \left(p \mathbf{v} \cdot \mathbf{n} - \frac{1}{\text{Re}} \mathbf{v} \cdot \nabla \mathbf{u} \cdot \mathbf{n} \right) dS = \frac{1}{\text{Re}} \int_{\Gamma_{s,t}} \mathbf{v} \cdot (\nabla \mathbf{u})^t \cdot \mathbf{n} \, dS$$



Weak formulation

Boundary term (II)

On Γ_b we use the free-slip (or Navier) boundary condition,

$$\begin{aligned}\int_{\Gamma_b} \left(p \mathbf{v} \cdot \mathbf{n} - \frac{1}{\text{Re}} \mathbf{v} \cdot \nabla \mathbf{u} \cdot \mathbf{n} \right) dS &= -\frac{1}{\text{Re}} \int_{\Gamma_b} \left([(\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + (\mathbf{v} \cdot \mathbf{t}) \mathbf{t}] \cdot \nabla \mathbf{u} \cdot \mathbf{n} \right) dS \\ &= -\frac{1}{\text{Re}} \int_{\Gamma_b} \left((\mathbf{v} \cdot \mathbf{t}) \mathbf{t} \cdot \nabla \mathbf{u} \cdot \mathbf{n} \right) dS \\ &= +\frac{1}{\text{Re}} \int_{\Gamma_b} \left((\mathbf{v} \cdot \mathbf{t}) \mathbf{t} \cdot (\nabla \mathbf{u})^t \cdot \mathbf{n} \right) dS \\ &= \frac{1}{\text{Re}} \int_{\Gamma_b} \left([(\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + (\mathbf{v} \cdot \mathbf{t}) \mathbf{t}] \cdot (\nabla \mathbf{u})^t \cdot \mathbf{n} \right) dS \\ &= \frac{1}{\text{Re}} \int_{\Gamma_b} \left(\mathbf{v} \cdot (\nabla \mathbf{u})^t \cdot \mathbf{n} \right) dS\end{aligned}$$

Weak formulation

Boundary term (III)

To summarize,

$$\int_{\partial\Omega_t} \left(p \mathbf{v} \cdot \mathbf{n} - \frac{1}{\text{Re}} \mathbf{v} \cdot \nabla \mathbf{u} \cdot \mathbf{n} \right) dS = \frac{1}{\text{Re}} \int_{\partial\Omega_t} \left(\mathbf{v} \cdot (\nabla \mathbf{u})^t \cdot \mathbf{n} \right) dS = \frac{1}{\text{Re}} \int_{\Omega_t} \nabla \mathbf{v} : (\nabla \mathbf{u})^t$$

so that

$$\begin{aligned} \int_{\Omega_t} \left(\mathbf{v} \cdot \nabla p - \frac{1}{\text{Re}} \mathbf{v} \cdot \Delta \mathbf{u} \right) &= \int_{\Omega_t} \left(\frac{1}{\text{Re}} \nabla \mathbf{v} : \nabla \mathbf{u} + \frac{1}{\text{Re}} \nabla \mathbf{v} : (\nabla \mathbf{u})^t - p \nabla \cdot \mathbf{v} \right) \\ &= \int_{\Omega_t} \left(\frac{2}{\text{Re}} \nabla \mathbf{v} : \mathbb{S}(\mathbf{u}) - p \nabla \cdot \mathbf{v} \right) \\ &= \int_{\Omega_t} \left(\frac{2}{\text{Re}} \mathbb{S}(\mathbf{v}) : \mathbb{S}(\mathbf{u}) - p \nabla \cdot \mathbf{v} \right) \end{aligned}$$

where $\mathbb{S}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^t)$.

Weak formulation

Final form

Find $\mathbf{u} \in \mathcal{C}^1([0, T]; \mathbf{H}_{\Gamma_b}^1(\Omega_t))$ and $p \in L^\infty([0, T], L^2(\Omega_t))$ such that

$$\int_{\Omega_t} \mathbf{v} \cdot \partial_t \mathbf{u} + \mathbf{v} \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{2}{\text{Re}} \mathbb{S}(\mathbf{v}) : \mathbb{S}(\mathbf{u}) - p \nabla \cdot \mathbf{v} + q \nabla \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{g} = 0$$

for all $\mathbf{v} \in \mathbf{H}_{\Gamma_b}^1(\Omega_t)$ and $q \in L^2(\Omega_t)$, at all time $t \in (0, T)$.

Intermezzo: Surface tension

Let Bo the Bond number,

$$\text{Bo} = \frac{\rho g L^2}{\sigma}$$

and $\kappa : [0, T) \times \Gamma_{s,t} \rightarrow \mathbb{R}$ the surface curvature.

Find $\mathbf{u} \in \mathcal{C}^1([0, T); \mathbf{H}_{\Gamma_b}^1(\Omega_t))$ and $p \in L^\infty([0, T), L^2(\Omega_t))$ such that

$$\int_{\Omega_t} \mathbf{v} \cdot \partial_t \mathbf{u} + \mathbf{v} \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{2}{\text{Re}} \mathbb{S}(\mathbf{v}) : \mathbb{S}(\mathbf{u}) - p \nabla \cdot \mathbf{v} + q \nabla \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{g} = \int_{\Gamma_{s,t}} \kappa \text{Bo}^{-1} \mathbf{v} \cdot \mathbf{n} \, dS$$

for all $\mathbf{v} \in \mathbf{H}_{\Gamma_b}^1(\Omega_t)$ and $q \in L^2(\Omega_t)$, at all time $t \in (0, T)$.

Numerical framework

Finite elements

Lagrangian advection

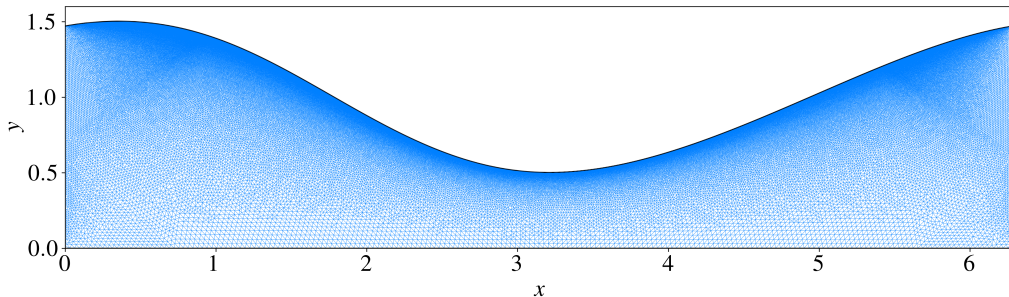
Geometric multigrid solver



Finite Elements discretization

We use the FreeFem finite elements library ([Hecht, 2012](#)) for

- Mesh generation and handling
- Matrices computations and handling
- Interface with PETSc



4 000 points on the interface, initially $\approx 200\,000$ triangles, $\approx 10^6$ degrees of freedom.

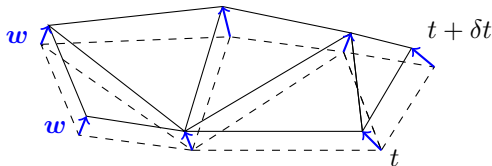
Mesh advection scheme

Let w the **mesh velocity**. At each time step, we numerically solve the problem

$$\begin{cases} \Delta w &= 0 & \text{in } \Omega_t \\ w &= u & \text{on } \Gamma_{s,t} \\ w &= 0 & \text{on } \Gamma_b \end{cases}$$

And each point of the mesh is advected according to w . Points on the interface are thus purely **Lagrangian**!

This is called the **Arbitrary Lagrangian Eulerian** method (ALE).



Time stepping

Crank-Nicolson second order in time scheme. **CFL** condition to compute the time step at each iteration.

At each time iteration, we solve the following weak problem

$$\begin{aligned} \int_{\Omega_t} \mathbf{v} \cdot \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t} + \mathbf{v} \cdot \left((\mathbf{u}^n - \mathbf{w}^n) \cdot \nabla \right) \left(\frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2} \right) + \frac{2}{\text{Re}} \mathbb{S}(\mathbf{v}) : \mathbb{S} \left(\frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2} \right) \\ + \left(\frac{p^{n+1} + p^n}{2} \right) \nabla \cdot \mathbf{v} + q \nabla \cdot \left(\frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2} \right) - \mathbf{v} \cdot \mathbf{g} \\ = 0 \end{aligned}$$

And then compute $\mathbf{w}^{n+1}$ before advecting the mesh.

At some time (typically at each output) we must recompute the mesh to prevent **triangle elongation**.

Parallelization and timings

MPI domain decomposition with graph partitionner. PETSc matrices and solvers.

We use a geometric multigrid solver for fast convergence using a large number of MPI processes.

At $Re = 10^6$ (between 1 and 3 million unknowns), convergence in $\sim 5 \pm 2$ GMRES iterations $\rightarrow \sim 20$ seconds on 48 CPU cores.

Main computational limitations due to FreeFem memory management and spurious behaviors in mesh handling.

From 3 days ($Re = 10^2$, 3 500 points on the interface) to 2 weeks of computations ($Re = 10^6$, 4 000 points on the interface) for one 2d simulation.

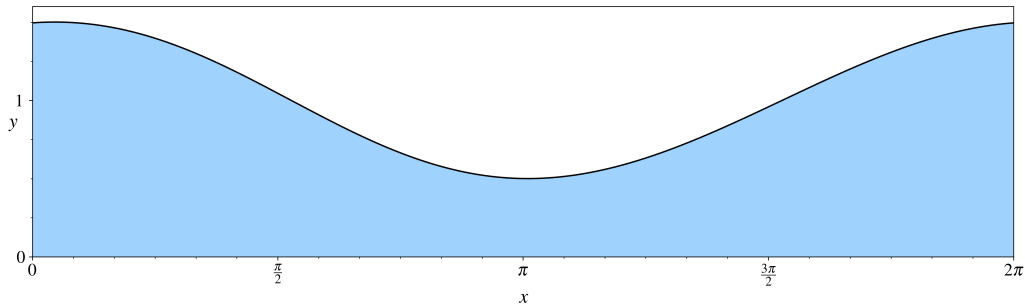
Clearly this method is not efficient enough to handle 3d, unless we decrease the precision.

The $\text{Re} \rightarrow +\infty$ limit

Comparing our results with the
non-viscous solution

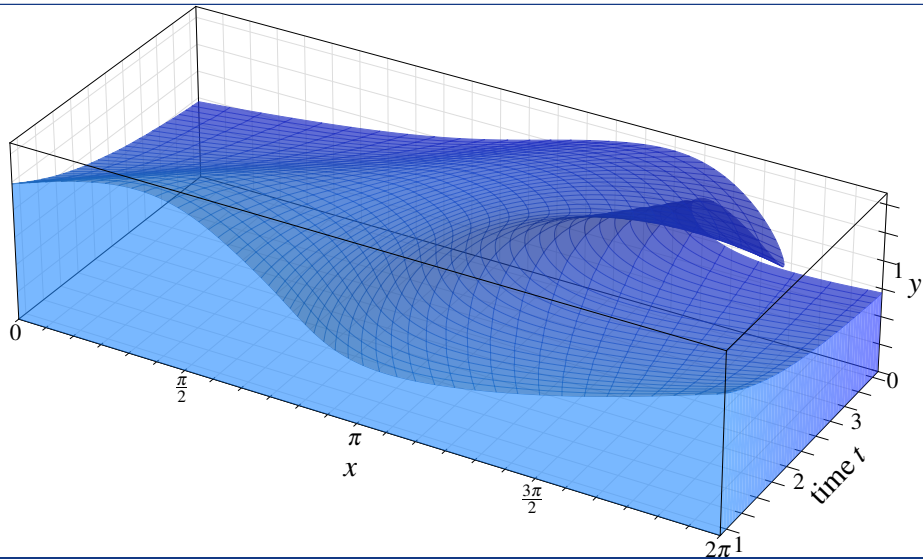


$\text{Re} = 10^6$ **result**

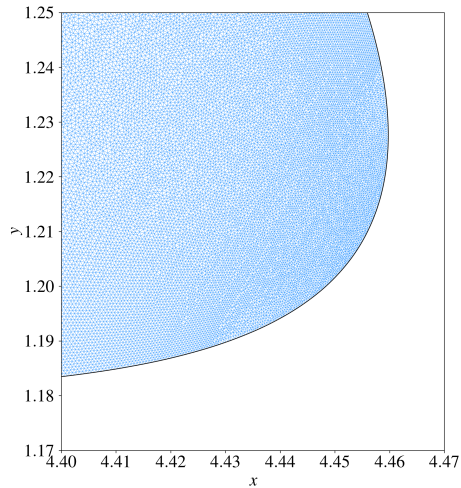
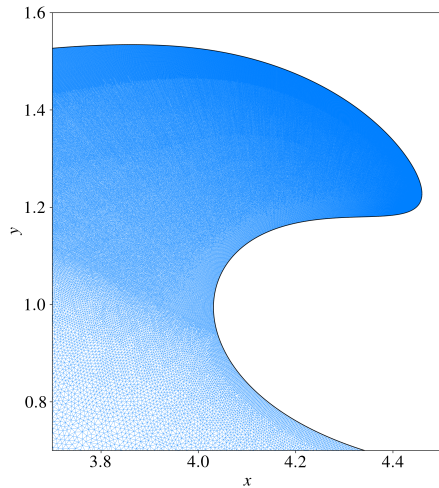


If the video does not play, click [here](#).

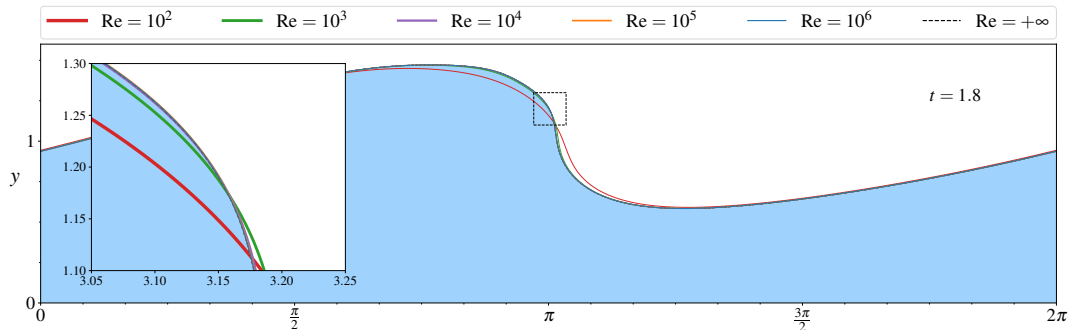
$Re = 10^6$ **result**



Mesh at $Re = 10^6$

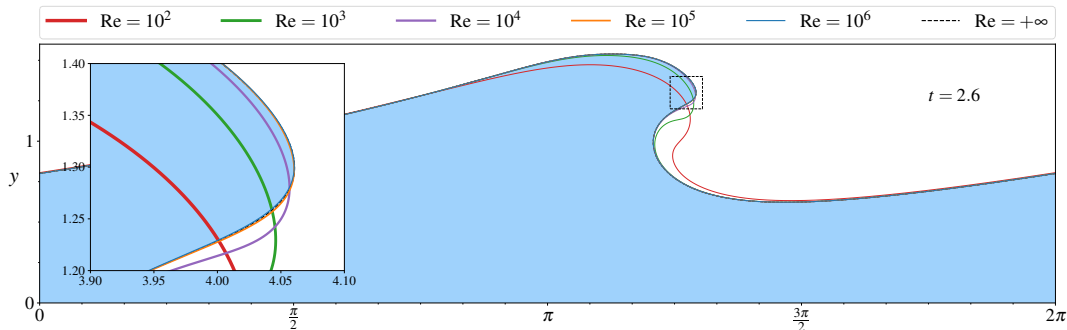


Interface for different values of Re



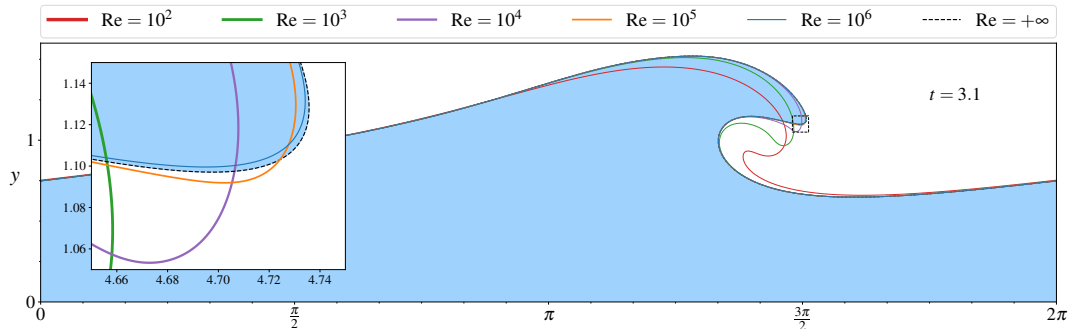
$Re = +\infty$ simulations (i.e. Euler solution) obtained using the code of [Dormy & Lacave \(2023\)](#).

Interface for different values of Re



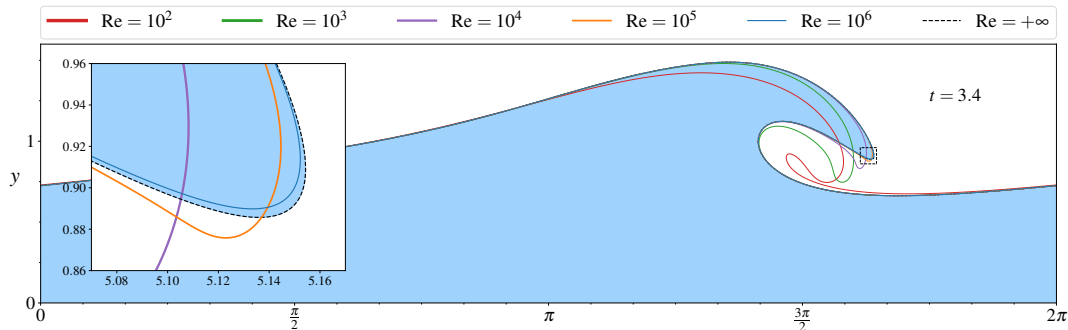
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Interface for different values of Re



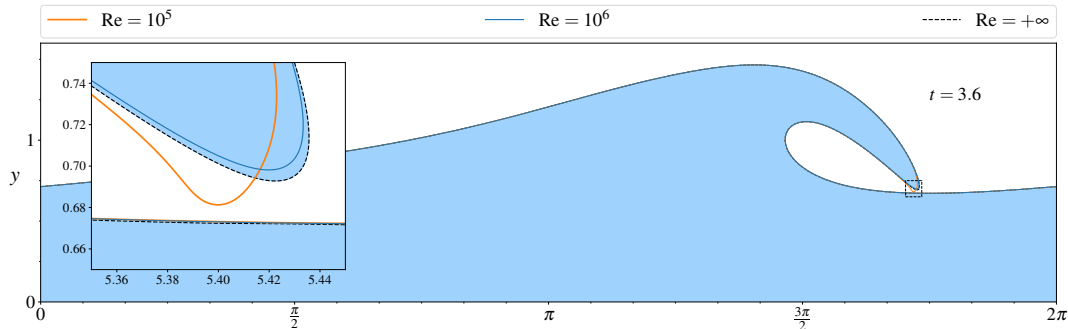
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Interface for different values of Re



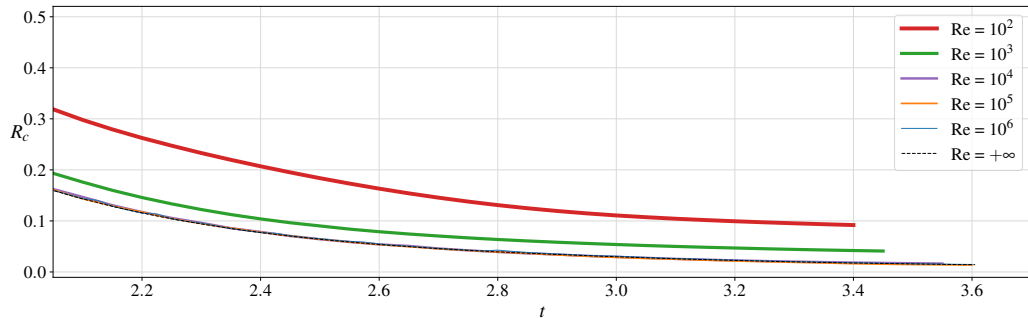
$Re = +\infty$ simulations (i.e. Euler solution) obtained using the code of [Dormy & Lacave \(2023\)](#).

Interface for different values of Re



$Re = +\infty$ simulations (i.e. Euler solution) obtained using the code of [Dormy & Lacave \(2023\)](#).

Maximum curvature of the interface



where $R_C = \kappa^{-1}$ is the curvature radius.

Boundary layer

Where does the viscous dissipation happen?



Energy considerations

Local and global

A **global** equation for the evolution of the energy can be obtained from the weak form of the viscous water waves problem, setting $v = \mathbf{u}(t, \cdot)$,

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega_t} \mathbf{u}^2 = \int_{\Omega_t} \mathbf{g} \cdot \mathbf{u} - \frac{2}{\text{Re}} \mathbb{S}(\mathbf{u}) : \mathbb{S}(\mathbf{u})$$

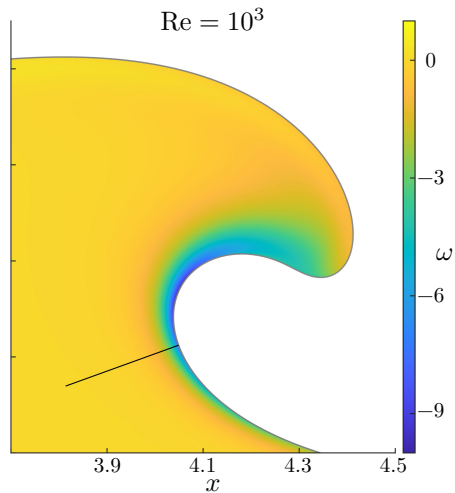
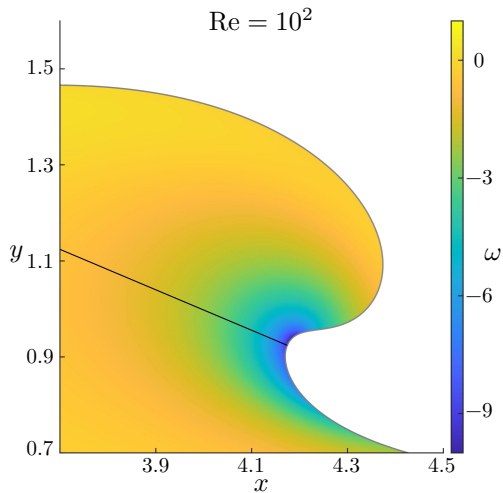
However this does not provide any knowledge about **where** this dissipation happens...

Instead, we work with the **local** energy equation, obtained multiplying the Navier-Stokes eq. by $\mathbf{u}$,

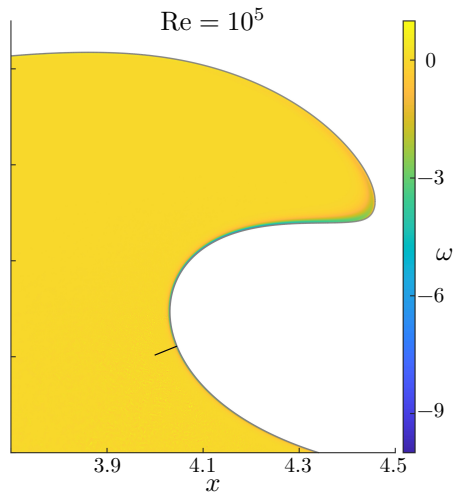
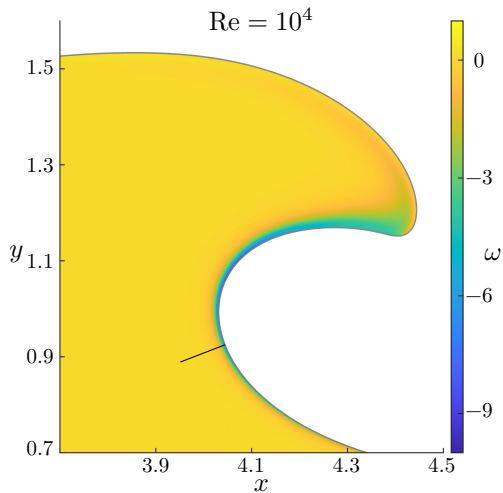
$$\partial_t \left(\frac{\mathbf{u}^2}{2} \right) = \mathbf{g} \cdot \mathbf{u} - \mathbf{u} \cdot \nabla p + \frac{1}{\text{Re}} \left[\nabla \cdot (\mathbf{u}^\perp \omega) - \omega^2 \right]$$

where $\mathbf{u}^\perp = [-u_y, u_x]$ and $\omega = \nabla^\perp \cdot \mathbf{u}$ is the **vorticity**.

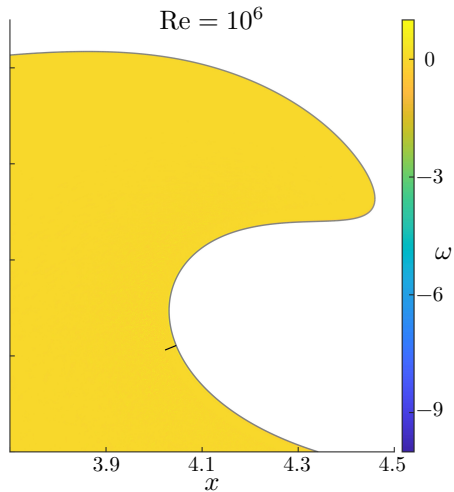
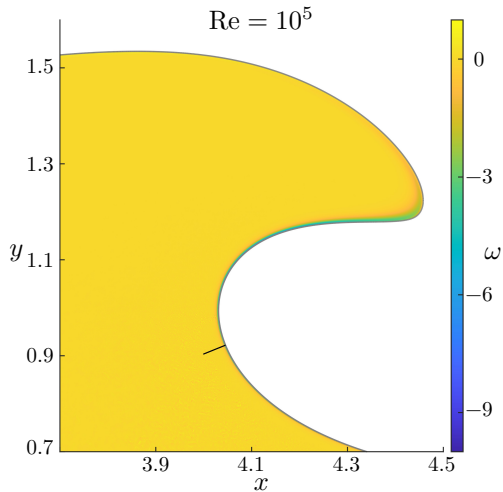
Viscous dissipation



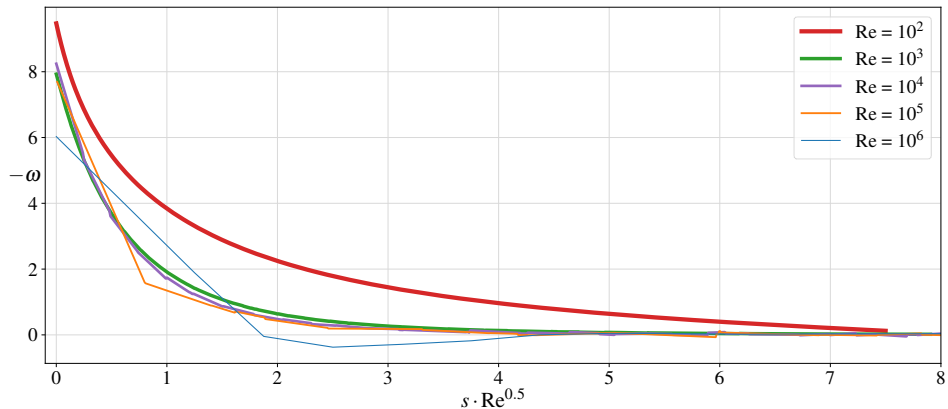
Viscous dissipation



Viscous dissipation



Size of the boundary layer



Exhibits a $\text{Re}^{-\frac{1}{2}}$ scaling (usual in BL theory).

Where does this Boundary Layer come from?

The superficial vorticity

We have encountered $\omega = \nabla^\perp \cdot \mathbf{u}$ the **volume** vorticity of the fluid. We can also make sense of the **superficial** vorticity γ whose definition is made through harmonic theory, as follows.

Let $G(\mathbf{x}, \mathbf{y})$ the **Green's function** of the Laplace operator Δ in a domain $\Omega \subset \mathbb{R}^2$. Then the function

$$\psi(\mathbf{x}) = \int_{\Omega} G(\mathbf{x}, \mathbf{y}) \omega(\mathbf{y}) \, d\mathbf{y}$$

is a solution of the Laplace equation $\Delta\psi = \omega$ in Ω .

If ω is the vorticity, ψ is the **stream function** such that $\mathbf{u} = \nabla^\perp \psi$. But ω can also be a distribution of the form $\gamma \delta_\Gamma$ where Γ is a curve in Ω . In that case,

$$\psi(\mathbf{x}) = \int_{\Gamma} G(\mathbf{x}, \mathbf{y}(s)) \gamma(\mathbf{y}(s)) \, ds$$

with $\mathbf{y}(s)$ a parametrization of the curve Γ . In general, γ depends on the parametrization. It is sometimes called the **vortex sheet strength**.

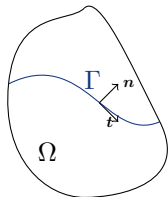
Where does this Boundary Layer come from?

The Lundgren-Koumoutsakos theorem

It can also be defined as

$$\gamma(s) = \left[\lim_{z \rightarrow 0^+} \mathbf{u}(\mathbf{y}(s) + z\mathbf{n}(s)) - \lim_{z \rightarrow 0^-} \mathbf{u}(\mathbf{y}(s) + z\mathbf{n}(s)) \right] \cdot \mathbf{t}(s)$$

if $\mathbf{y}(s)$ is the arc-length parametrization.



The Lundgren & Koumoutsakos (1999) theorem states that if the vorticity is composed of a superficial part γ and a volume part ω , the former diffuses into the latter due to viscous effects

$$\frac{d}{dt} \int_{\Gamma} \gamma dS + \int_{\Gamma} \frac{1}{\text{Re}} \nabla \omega \cdot \mathbf{n} dS = 0$$

in a way that the total vorticity is conserved,

$$\frac{d}{dt} \int_{\Gamma} \gamma dS + \frac{d}{dt} \int_{\Omega} \omega dV = 0$$

FYI: The Vortex method

If we assimilate the plane $\mathbb{R}^2$ as the complex plane $\mathbb{C}$, we can rewrite this problem in **complex variables**,

$$\boldsymbol{x} \rightarrow z = x + iy \qquad \boldsymbol{u} \rightarrow u = u_x - iu_y$$

In that case the velocity u can be related to the (**complex**) vortex sheet strength γ as

$$u(z) = \frac{1}{2iL} \int_0^L \cot \left(\pi \frac{z - z(s)}{L} \right) \gamma(s) \, ds$$

where L is the domain length and $z(s)$ is a parametrization of the surface.

From the **Bernoulli's eq.** we are able to find a law for the evolution of γ , namely

$$\partial_t \gamma + \partial_t \left(\partial_s z \cdot u_{\text{air}}[z(s)] \right) - \left(\partial_{ts} z \cdot u_{\text{water}}[z(s)] \right) = -g \partial_s y$$

This is only valid in the **deep water** limit. If we consider the **bottom**, there is also an equation for the bottom sheet γ_b .

More informations: [Dormy & Lacave \(2023\)](#).

Viscosity and free surfaces

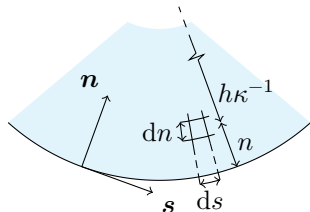
A general theory of interface
regularization due to viscosity



An equation for the Curvature evolution

In Longuet-Higgins (1953) is derived an equation for the curvature evolution, provided the interface is a $\mathcal{C}^3$ function.

Introduce the Frenet frame (s, n) , i.e. the tangential and normal coordinate w.r.t. the (time-dependant) curve $\gamma(t, s)$.



A point $\gamma(t, s)$ on the curve is advected by a vector field $\mathbf{u}(s, n) = (u_s, u_n)$.

The curvature $\kappa(t, s)$ obeys

$$\partial_t \kappa = \partial_{ss} u_n - u_s \partial_s \kappa + \kappa^2 u_n$$

The derivation of this PDE is geometric.

Stream function

Expressing the **incompressibility condition** in (s, n) coordinates yields

$$\nabla \cdot \mathbf{u} = \partial_s u_s + \partial_n (h u_n) = 0 \quad \text{with} \quad h = 1 - \kappa^{-1} n$$

This motivates the definition of a **stream function** ψ s.t.

$$u_s = -\partial_n \psi \quad \text{and} \quad u_n = h^{-1} \partial_s \psi$$

The vorticity is

$$\omega = h^{-1} \left(\partial_s u_n - \partial_n (h u_s) \right) = \Delta \psi$$

We **suppose** that the velocity $\mathbf{u}$ can be expressed as an irrotational part and a viscous part

$$\mathbf{u} = \nabla \phi + \nabla^\perp \psi_{\text{Re}}$$

Asymptotic expansion

We observed that the **typical size** δ of the boundary layer is

$$\delta = \frac{1}{\sqrt{\text{Re}}}$$

We therefore suppose that the deppendance of ψ in the viscosity Re is of the form

$$\psi_{\text{Re}}(t, s, n) = \psi_0(t, s, n) + \delta\psi_1(t, s, n) + \delta^2\psi_2(t, s, n) + \mathcal{O}(\delta^3)$$

Since the viscous effects seem to vanish as $\text{Re} \rightarrow +\infty$, we get $\psi_0 = 0$.

The typical variation length of ψ in the normal direction is δ . Hence we can assume that

$$\psi_{\text{Re}}(t, s, n) \equiv \psi_{\text{Re}}(t, s, n\delta^{-1})$$

so **each normal derivative decreases the order of the expansion by 1.**

Effects of viscosity on the curvature

Inserting the expansion in the vorticity equation, we find that the **leading term** is of **order** $\mathcal{O}(\delta^{-1})$. However we **observed** that the vorticity seem to behave as $\mathcal{O}(1)$. Therefore $\psi_1 = 0$.

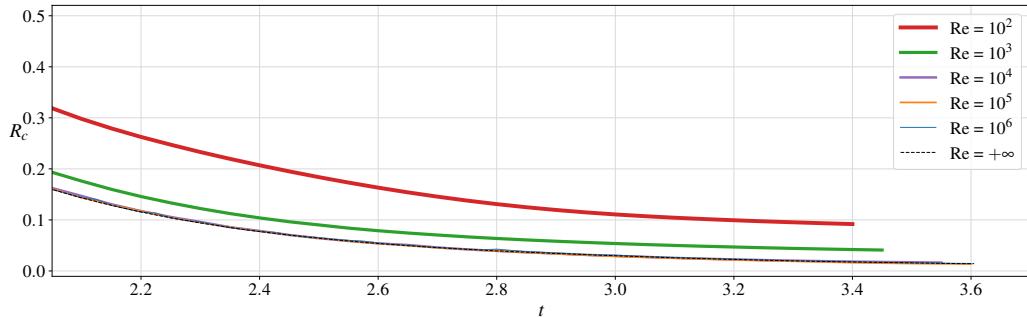
Inserting the expansion in the equation for κ , we get

$$\partial_t \kappa = \text{irrotational part} + \underbrace{\partial_{ss}(h^{-1}\partial_s\psi_{\text{Re}})}_{\mathcal{O}(\delta^2)} - \underbrace{\partial_s\kappa\partial_n\psi_{\text{Re}}}_{\mathcal{O}(\kappa\delta)} + \underbrace{\kappa^2 h^{-1}\partial_s\psi_{\text{Re}}}_{\mathcal{O}(\kappa^2\delta^2)}$$

Interpretation:

- The effects of viscosity appear in time $\mathcal{O}(1)$ when the curvature is of order $\mathcal{O}(\delta^{-1})$
- The effects of viscosity appear in time $\mathcal{O}(\delta^{-2})$ when the curvature is of order $\mathcal{O}(\delta)$

Maximum curvature of the interface



where $R_C = \kappa^{-1}$ is the curvature radius.

Thank you!