



ON THE VANISHING VISCOSITY LIMIT IN WATER WAVES

CIMAV 2025 – Aussois 

Alan Riquier - Département de Mathématiques et Applications (ENS - PSL)

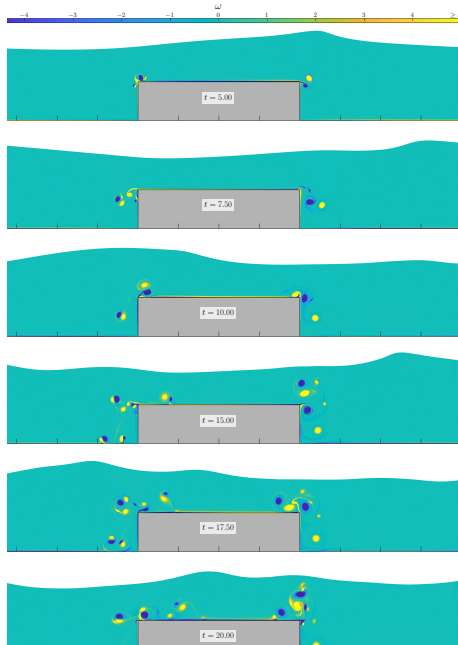
Joint work with **Emmanuel Dormy** (DMA - ENS PSL)

May 12, 2025



Outline

1. Mathematical Modeling
2. Numerical Methods
3. The Free-Surface Boundary Layer
4. Bottom Boundary Layers and Topography
5. Do initially irrotational Water Waves remain so?



Problem formulation

Navier-Stokes

Lagrangian advection

Initial condition



Viscous Water Waves

Nondimensionalisation

Nondimensional quantities are defined as follows

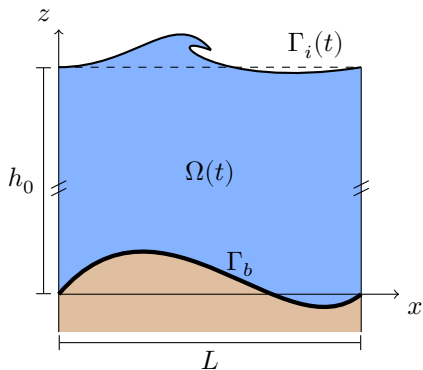
$$x \rightarrow h_0 x$$

$$\mathbf{u} \rightarrow \sqrt{gh_0} \cdot \mathbf{u}$$

$$p \rightarrow \rho gh_0 \cdot p$$

We obtain a **Reynolds number** Re ,

$$\text{Re} = \frac{\rho h_0 \sqrt{gh_0}}{\mu}$$



Viscous Water Waves

Navier problem

Incompressible, non-dimensional, **Navier-Stokes** equation in $\Omega(t)$:

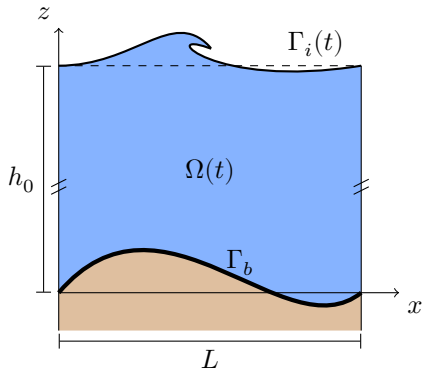
$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla p + \frac{1}{\text{Re}} \Delta \mathbf{u} + \mathbf{g} \\ \nabla \cdot \mathbf{u} &= 0 \end{cases}$$

Stress-free boundary condition on $\Gamma_i(t)$,

$$p\mathbf{n} - \frac{1}{\text{Re}} \cdot [\nabla \mathbf{u} + (\nabla \mathbf{u})^\top] \cdot \mathbf{n} = 0$$

slip/Navier boundary conditions on Γ_b ,

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad ; \quad t \cdot [\nabla \mathbf{u} + (\nabla \mathbf{u})^t] \cdot \mathbf{n} = 0$$



Viscous Water Waves

Dirichlet problem

Incompressible, non-dimensional, **Navier-Stokes** equation in $\Omega(t)$:

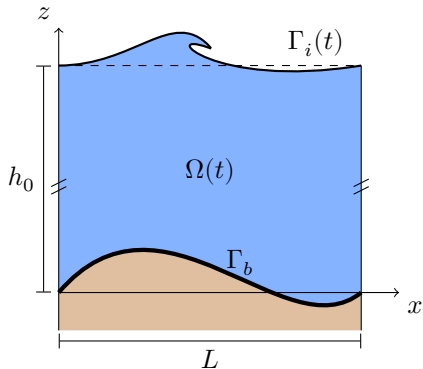
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no-slip/Dirichlet boundary conditions on Γ_b ,

$$\mathbf{u} = 0$$



Weak formulation

Navier-Stokes problem

Velocity $\mathbf{u}$ and **pressure** p such that

$$\int_{\Omega(t)} \mathbf{v} \cdot \partial_t \mathbf{u} + \mathbf{v} \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{2}{\text{Re}} \mathbb{S}(\mathbf{v}) : \mathbb{S}(\mathbf{u}) - p \nabla \cdot \mathbf{v} + q \nabla \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{g} = 0$$

for all $\mathbf{v}$ and q . (+ **no-slip/Dirichlet** or **slip/Navier** in the function space)

$\mathbb{S}(\mathbf{u})$ is the symmetric part of the gradient,

$$\mathbb{S}(\mathbf{u}) = \frac{1}{2} \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^t \right]$$

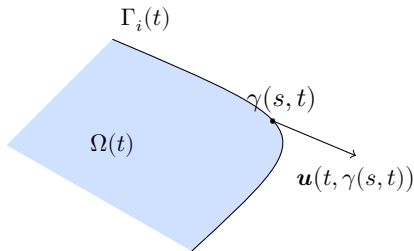
Interface advection

Lagrangian scheme

Interface is a **parametrised curve** $\gamma(s, t) \in \mathbb{R}^2$ whose evolution is given by

$$\frac{\partial \gamma}{\partial t}(s, t) = \mathbf{u}(t, \gamma(s, t))$$

i.e. **points on the interface** have the **same velocity as the fluid particles**.



The interface contains all points parametrised by γ ,

$$\Gamma_i(t) = \bigcup_s \{\gamma(s, t)\}$$

Initial conditions

$$\mathbf{u}_0(t, x, y) = \frac{a\omega}{\sinh(kh_0)} \cdot \begin{bmatrix} \cosh(kh_0) \cos(kx - \omega t) \\ \sinh(kh_0) \sin(kx - \omega t) \end{bmatrix}$$

so the velocity $\mathbf{n} \cdot \mathbf{u}_0$ in the **normal** direction can be computed from

$$\mathbf{n}_0(x) = \frac{1}{\sqrt{1 + (\partial_x \gamma_0)^2}} \begin{bmatrix} -\partial_x \gamma_0 \\ 1 \end{bmatrix} \quad \text{where} \quad \gamma_0(x) = h_0 + a \cos(kx)$$

The **initial velocity** is then constructed by solving the following Laplace problem

$$\begin{cases} \Delta \phi_0 &= 0 & \text{in the domain } \Omega(t) \\ \partial_n \phi_0 &= 0 & \text{on the bottom } \Gamma_b \\ \partial_n \phi_0 &= \mathbf{n}_0 \cdot \mathbf{u}_0(y = h_0) & \text{on the surface } \Gamma_i(t = 0) \end{cases}$$

and the **initial velocity** is $\mathbf{u}(t = 0) = \nabla \phi_0$.

Numerical Methods

Finite elements

Mixed Lagrangian-Eulerian

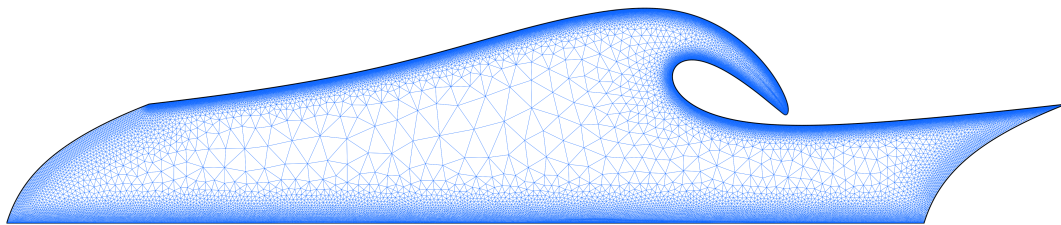
Geometric multigrid solver



Finite Elements discretization

We use FreeFEM for

- Mesh generation and handling
- Matrices computations and handling
- Interface with PETSc



4 000 points on the interface, initially $\approx 200\,000$ triangles, $\approx 10^6$ degrees of freedom.

Mesh advection scheme

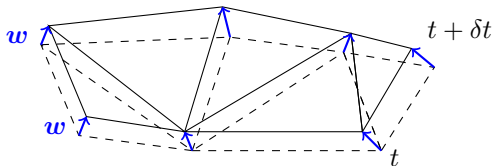
Elliptic method

Let w the **velocity of the mesh**. At each time step, we **numerically solve the problem**

$$\begin{cases} \Delta w &= 0 & \text{in } \Omega_t \\ w &= u & \text{on } \Gamma_{s,t} \\ w &= 0 & \text{on } \Gamma_b \end{cases}$$

And we use the **movemesh** function to **advect** the mesh with velocity w . **Points on the interface** are thus **purely Lagrangian**!

This is called the **Arbitrary Lagrangian Eulerian** method (ALE).

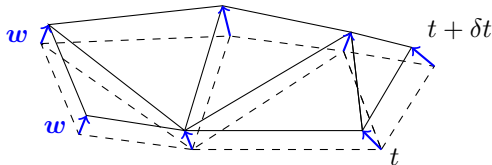


Mesh advection scheme

Fully Lagrangian method

Let w the **velocity of the mesh**. We can also set

$$w = u.$$



Time stepping scheme

Implicit Euler time scheme. **CFL** condition to compute the **time step** at each iteration.

At each time step, we solve the following problem

$$\begin{aligned} \int_{\Omega(t)} \mathbf{v} \cdot \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t} + \mathbf{v} \cdot \left((\mathbf{u}^n - \mathbf{w}^n) \cdot \nabla \right) \mathbf{u}^{n+1} + \frac{2}{\text{Re}} \mathbb{S}(\mathbf{v}) : \mathbb{S}(\mathbf{u}^{n+1}) \\ + p^{n+1} \nabla \cdot \mathbf{v} + q \nabla \cdot \mathbf{u}^{n+1} - \mathbf{v} \cdot \mathbf{g} \\ = 0 \end{aligned}$$

for all $(\mathbf{v}, q)$ and then **compute** $\mathbf{w}^{n+1}$ **before** advecting the mesh.

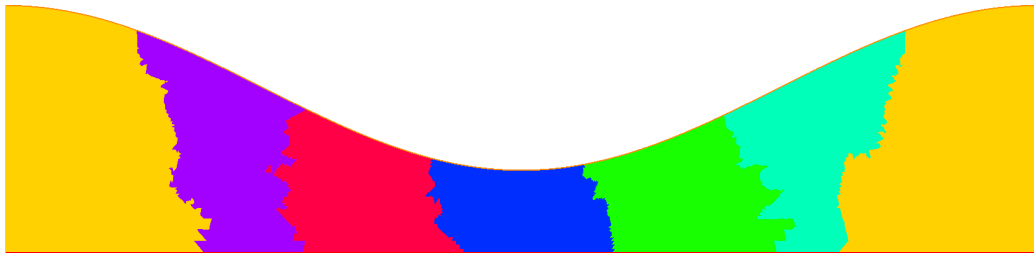
We use $\mathbb{P}^2$ **elements** for the velocity and $\mathbb{P}^1$ **elements** for the pressure.

Lagrangian advection $\mathbf{w}^n = \mathbf{u}^n \longrightarrow$ **No nonlinear term!**

Domain decomposition

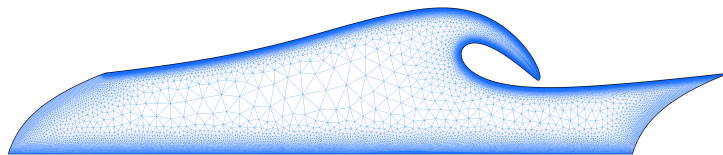
MPI domain decomposition with graph partitioner. **PETSc** matrices and solvers.

Example with **6 domains**:

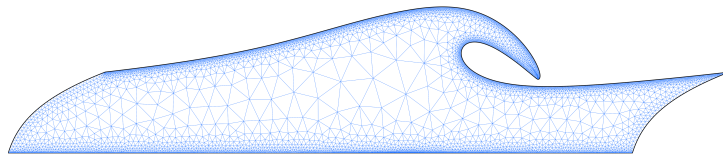


Non-nested Geometric multigrid

We use a **geometric multigrid** solver for fast convergence using a **large number of MPI processes**.

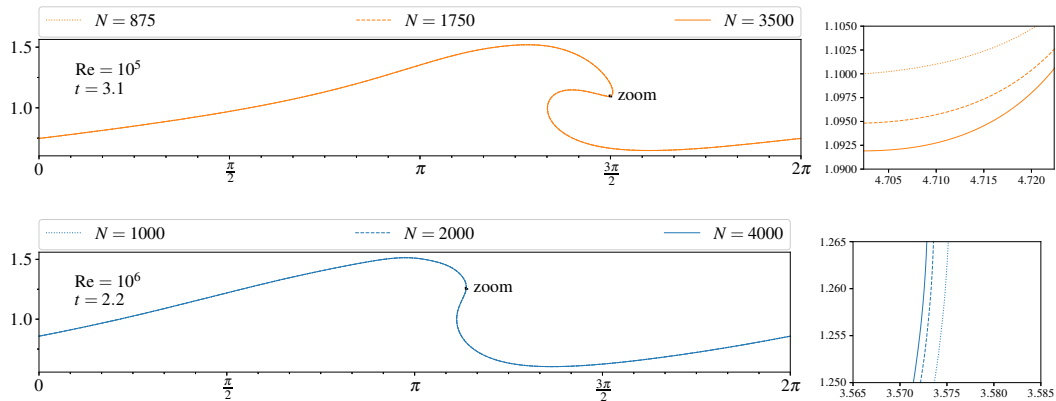


Level 0 (fine)



Level 2 (coarse)

Convergence



The $Re \rightarrow +\infty$ limit, part I

Surface Boundary Layer



Viscous Water Waves

Navier problem

Incompressible, non-dimensional, **Navier-Stokes** equation in $\Omega(t)$:

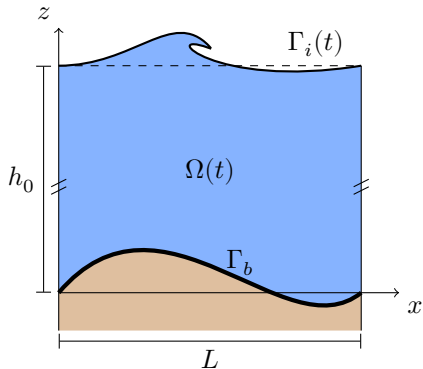
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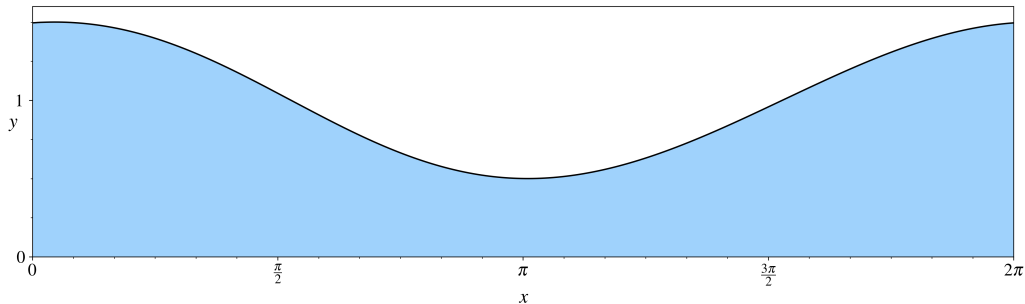
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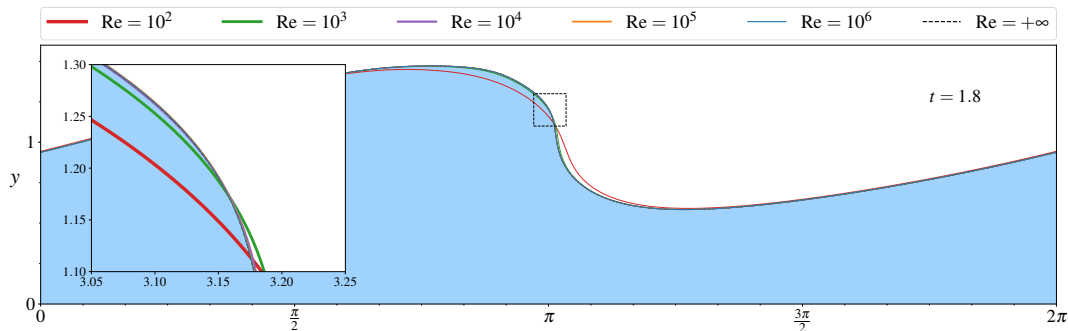


$\text{Re} = 10^6$ **result**



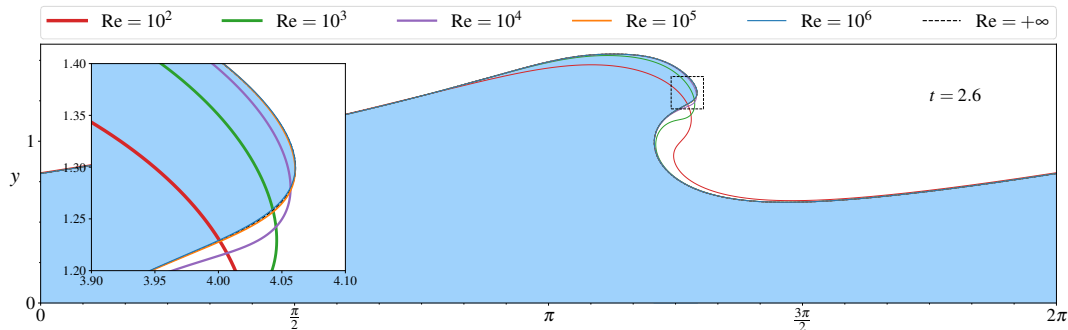
If the video does not play, click [here](#).

Interface for different values of Re



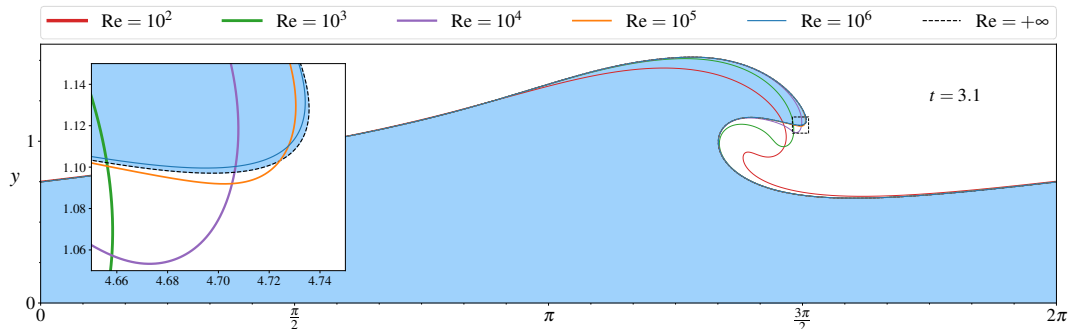
$Re = +\infty$ simulations (i.e. Euler's irrotational solution) computed with the numerical methods of [Dormy & Lacave \(2024\)](#).

Interface for different values of Re



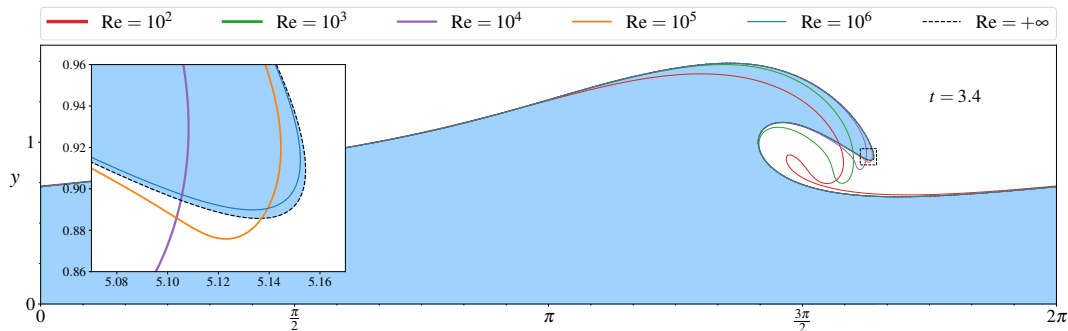
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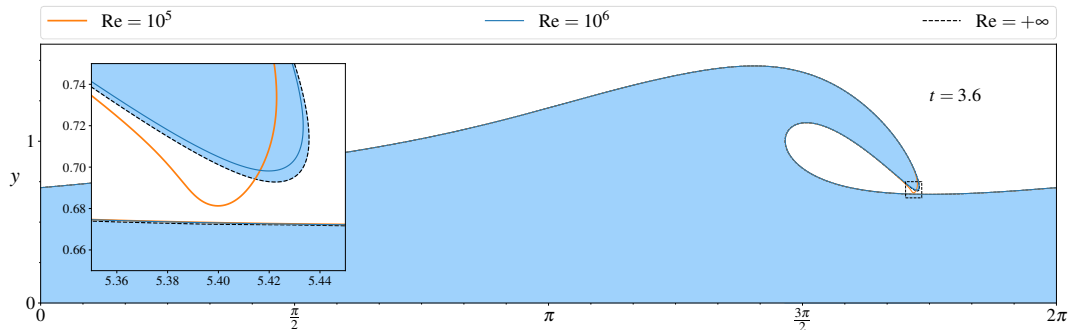
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Interface for different values of Re



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Surface boundary layer

Vorticity generation

Take the **Stress-Free** boundary condition,

$$pn - \frac{1}{\text{Re}} \cdot [\nabla \mathbf{u} + (\nabla \mathbf{u})^\top] \cdot \mathbf{n} = 0$$

Surface boundary layer

Vorticity generation

Take the **Stress-Free** boundary condition,

$$pn - \frac{1}{\text{Re}} \cdot [\nabla \mathbf{u} + (\nabla \mathbf{u})^\top] \cdot \mathbf{n} = 0$$

Keep the tangential part

$$\mathbf{t} \cdot [\nabla \mathbf{u} + (\nabla \mathbf{u})^\top] \cdot \mathbf{n} = 0$$

Surface boundary layer

Vorticity generation

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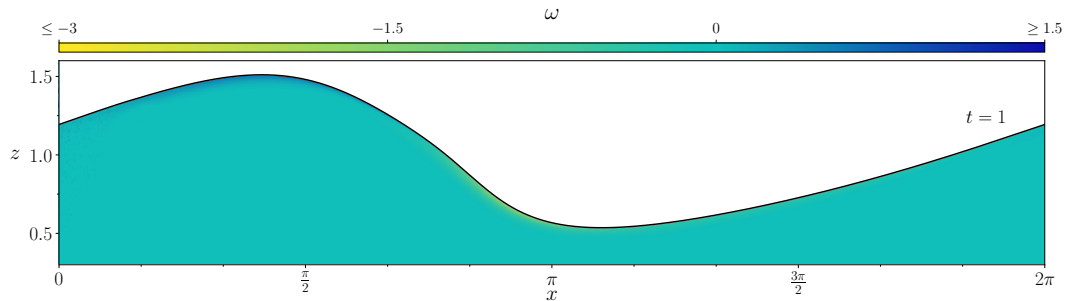
$$\mathbf{t} \cdot [\nabla \mathbf{u} + (\nabla \mathbf{u})^\top] \cdot \mathbf{n} = 0$$

Inserting this in the **vorticity** on the **free surface** yields

$$\begin{aligned}\omega &= \mathbf{n} \cdot \nabla \mathbf{u} \cdot \mathbf{t} - \mathbf{t} \cdot \nabla \mathbf{u} \cdot \mathbf{n} \\ &= -2\mathbf{t} \cdot \nabla \mathbf{u} \cdot \mathbf{n} \\ &= -2\partial_s \mathbf{u} \cdot \mathbf{n} \\ &= -2\partial_s (\mathbf{u} \cdot \mathbf{n}) + 2(\mathbf{u} \cdot \mathbf{t})\kappa\end{aligned}$$

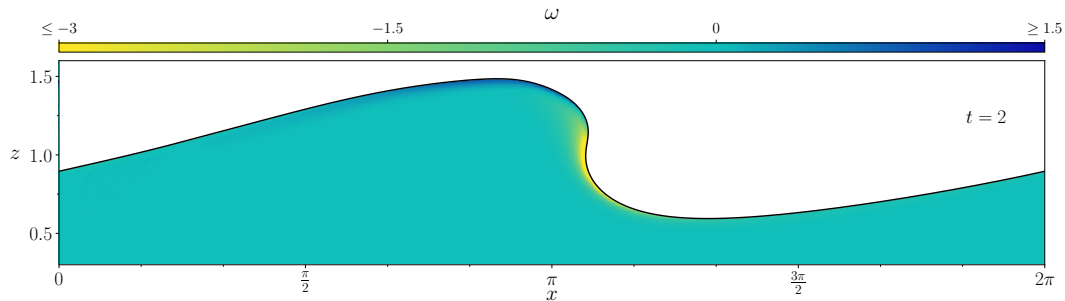
with κ the **curvature** of the free surface.

Evolution of the vorticity ω at $\text{Re} = 10^3$



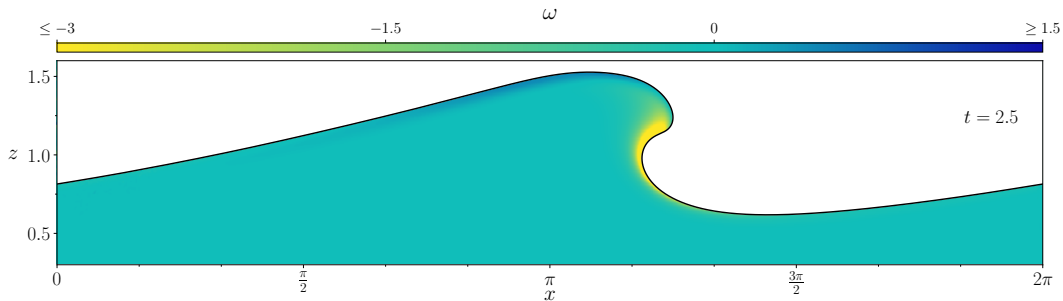
$$\omega = -2\partial_s(\mathbf{u} \cdot \mathbf{n}) + 2(\mathbf{u} \cdot \mathbf{t})\kappa$$

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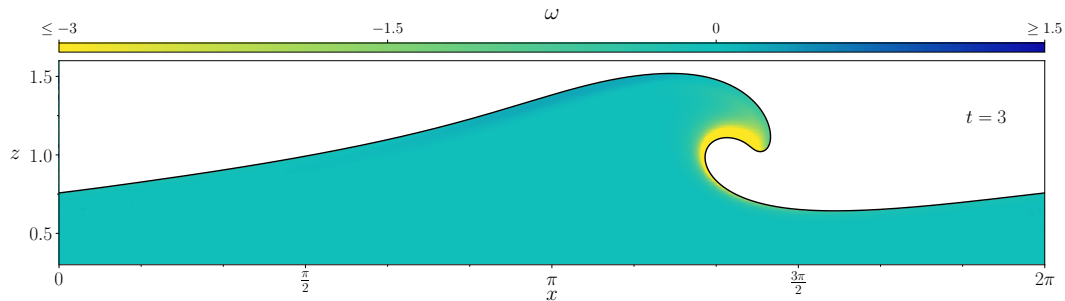
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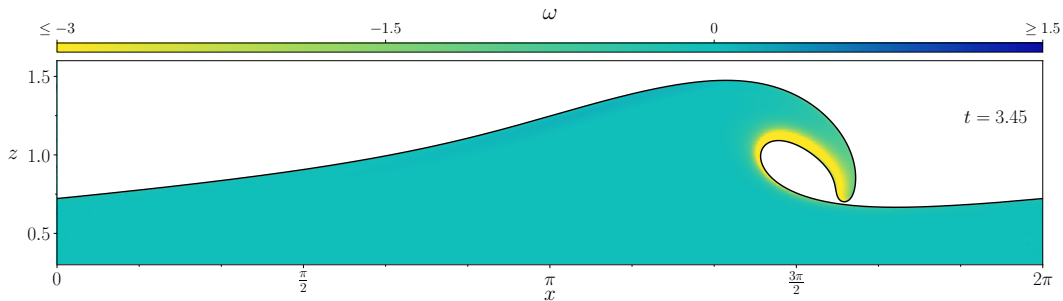
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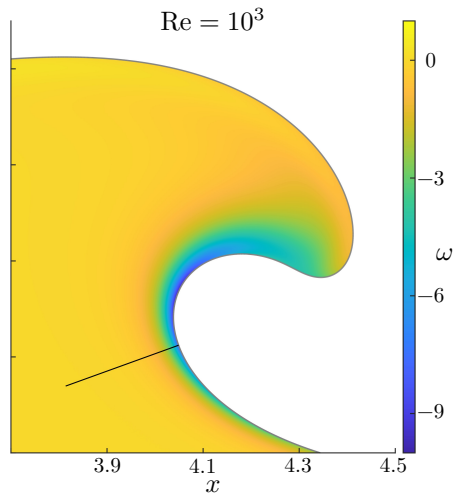
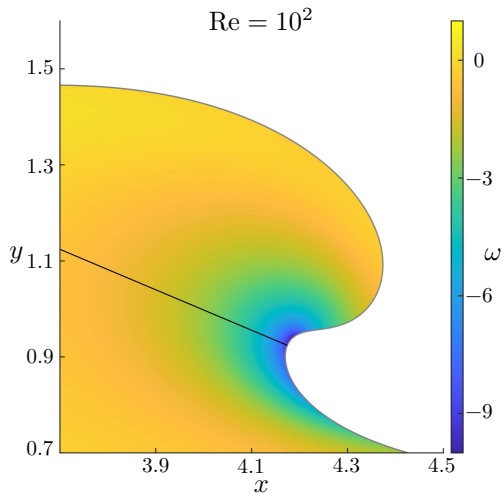
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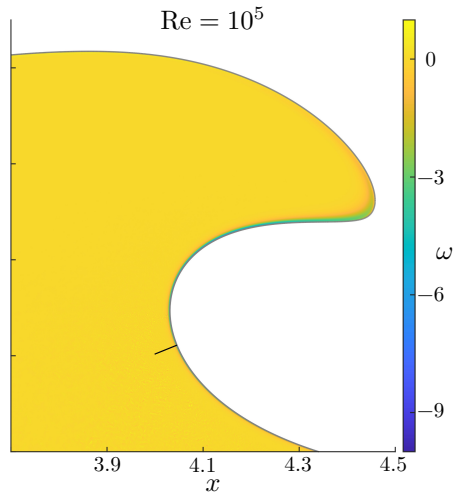
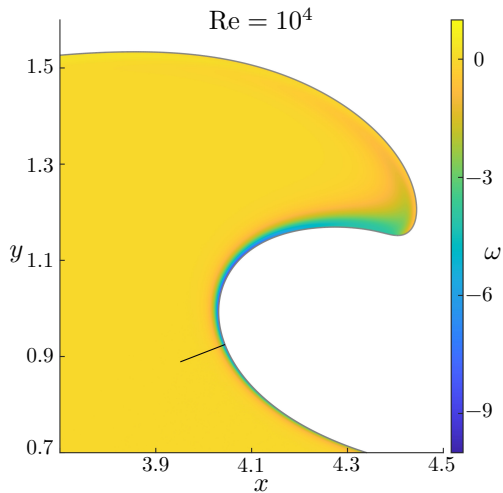


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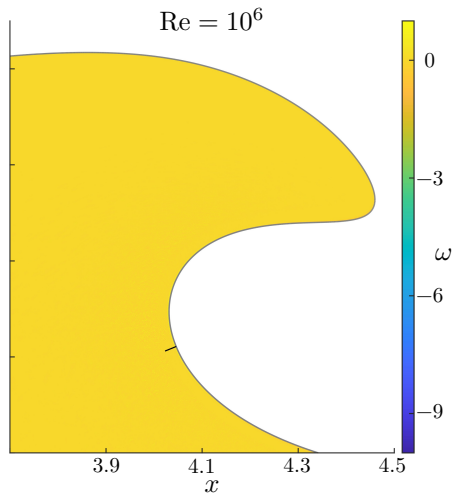
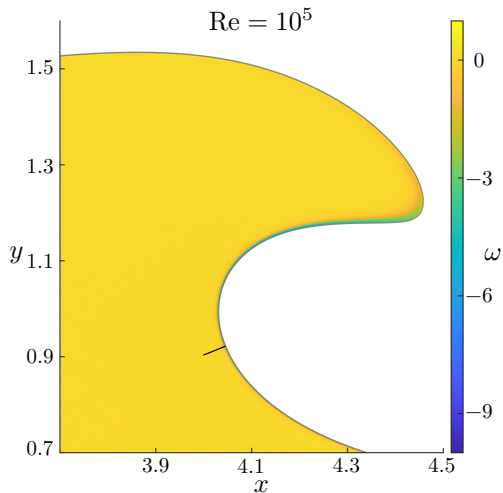
Boundary layer at fixed time $t = 2.9$



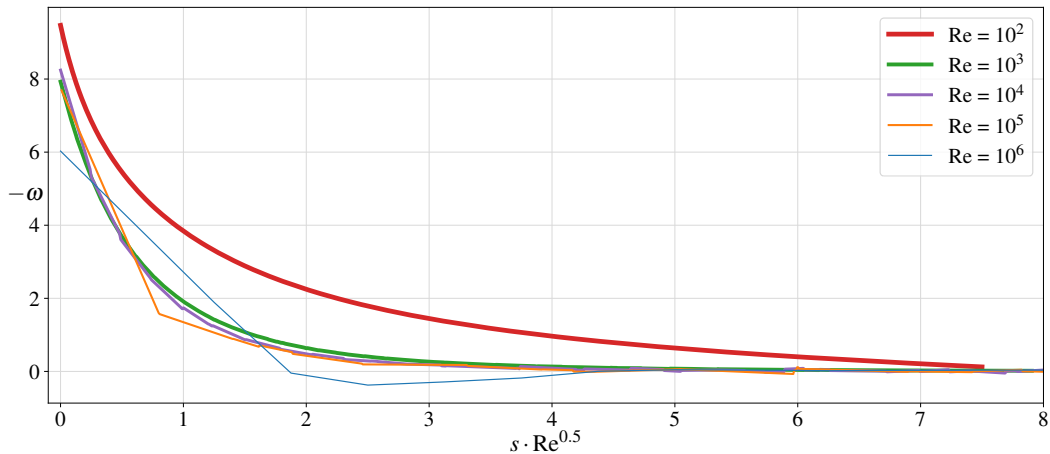
Boundary layer at fixed time $t = 2.9$



Boundary layer at fixed time $t = 2.9$



Scaling of the vorticity sheet



The $\text{Re} \rightarrow +\infty$ limit, part II

Bottom Boundary Layer



Viscous Water Waves

Dirichlet problem

Incompressible, non-dimensional, **Navier-Stokes** equation in $\Omega(t)$:

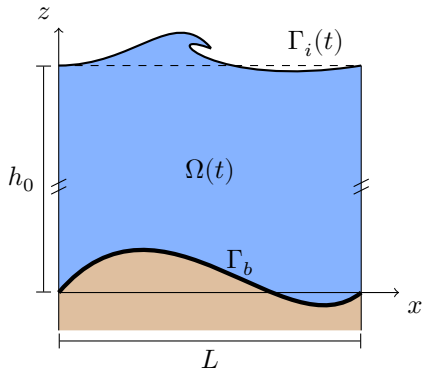
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no-slip/Dirichlet boundary conditions on Γ_b ,

$$\mathbf{u} = 0$$



Motivation I

MASS TRANSPORT IN WATER WAVES

By M. S. LONGUET-HIGGINS

Trinity College, University of Cambridge

(Communicated by R. Stoneley, F.R.S.—Received 14 October 1952—Read 19 February 1953)

Motivation I

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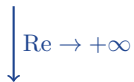
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~~... of vorticity, ... will be ...~~ In an oscillating motion this vorticity will be of alternating sign; and the question then presents itself: will any of the vorticity spread into the interior of the fluid, or will it remain in the neighbourhood of the boundary?

Motivation II

free-surface Navier-Stokes



free-surface Euler

Motivation II

free-surface Navier-Stokes

↓ $\text{Re} \rightarrow +\infty$

free-surface Euler

↓ the flow is initially irrotational

Water Waves

Motivation II

free-surface Navier-Stokes

↓ $\text{Re} \rightarrow +\infty$ → deep water:  Masmoudi & Rousset (2017)

free-surface Euler

↓ the flow is initially irrotational

Water Waves

Motivation II

free-surface Navier-Stokes



$\text{Re} \rightarrow +\infty$



deep water:  Masmoudi & Rousset (2017)
slip/Navier bottom: ?

free-surface Euler



the flow is initially irrotational

Water Waves

Motivation II

free-surface Navier-Stokes



$Re \rightarrow +\infty$



deep water:  Masmoudi & Rousset (2017)
slip/Navier bottom: ?
no-slip/Dirichlet bottom: ???

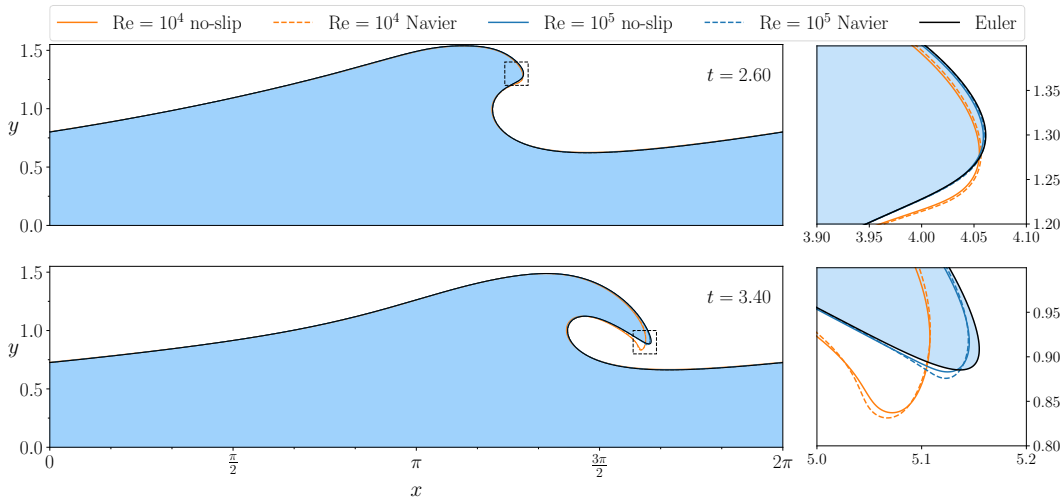
free-surface Euler



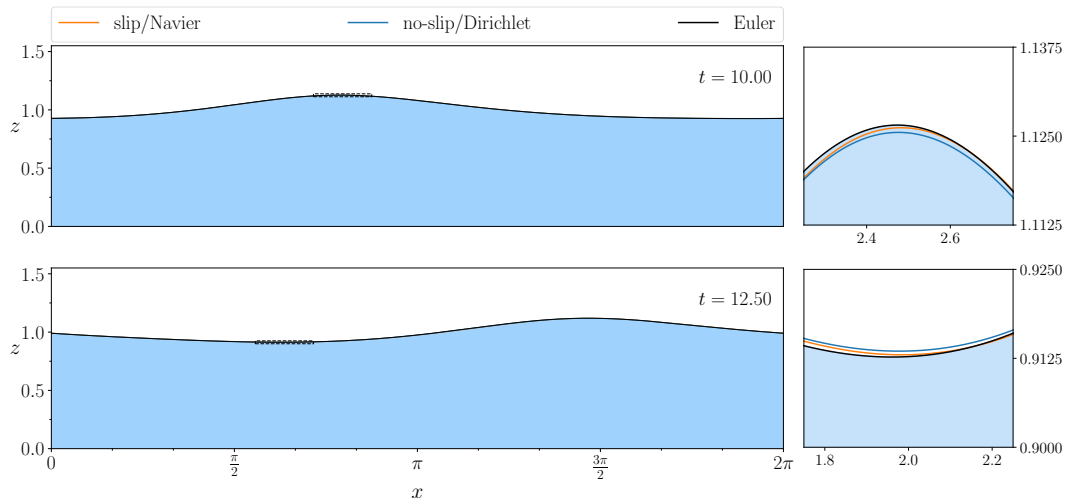
the flow is initially irrotational

Water Waves

Breaking Wave over a Flat Topography



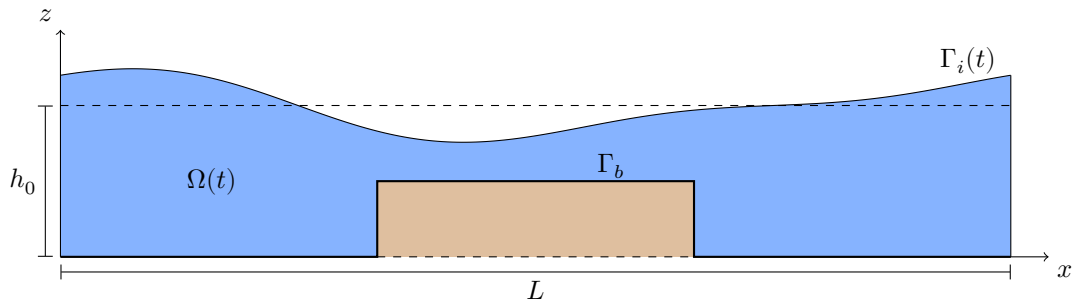
Non-Breaking Wave over a Flat Topography ($\text{Re} = 10^5$)



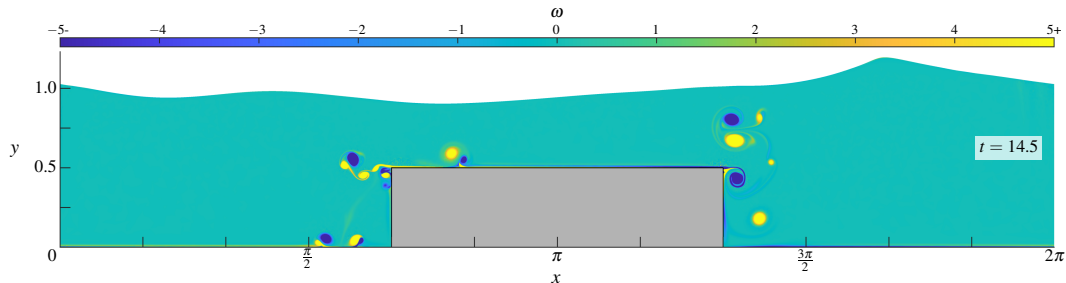
Rectangular step

No slip / **Dirichlet** condition on the bottom,

$$u = 0 \quad \text{on } \Gamma_b$$

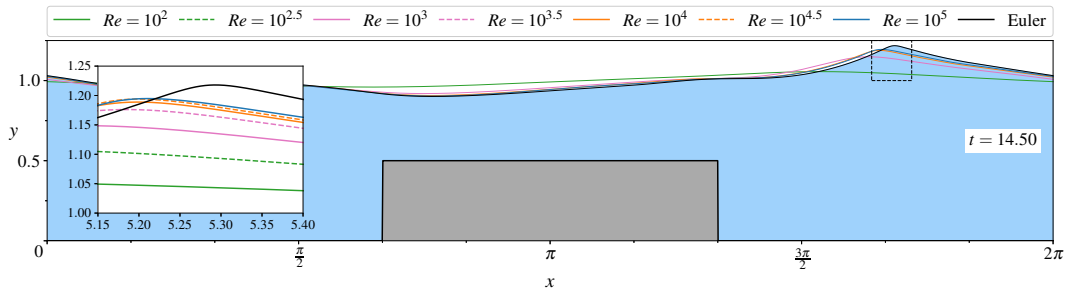


Vortices at $\text{Re} = 10^5$

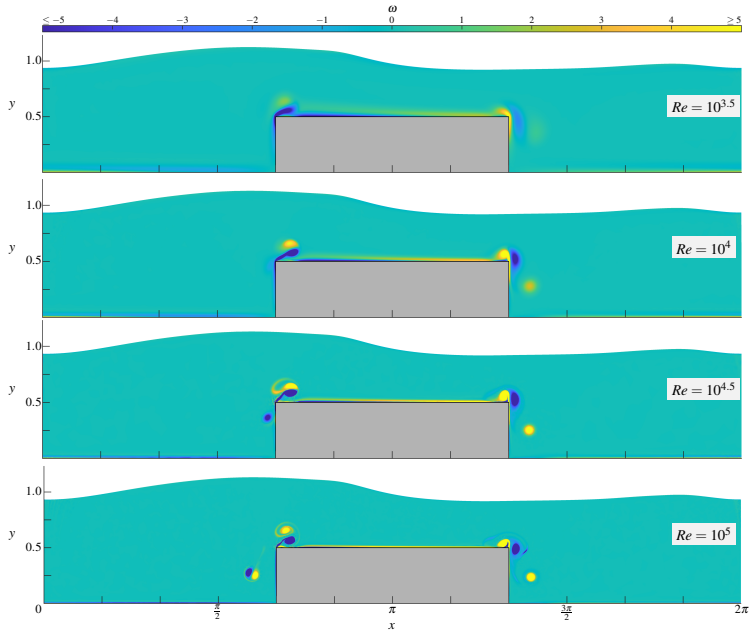


If the video does not play, click [here](#).

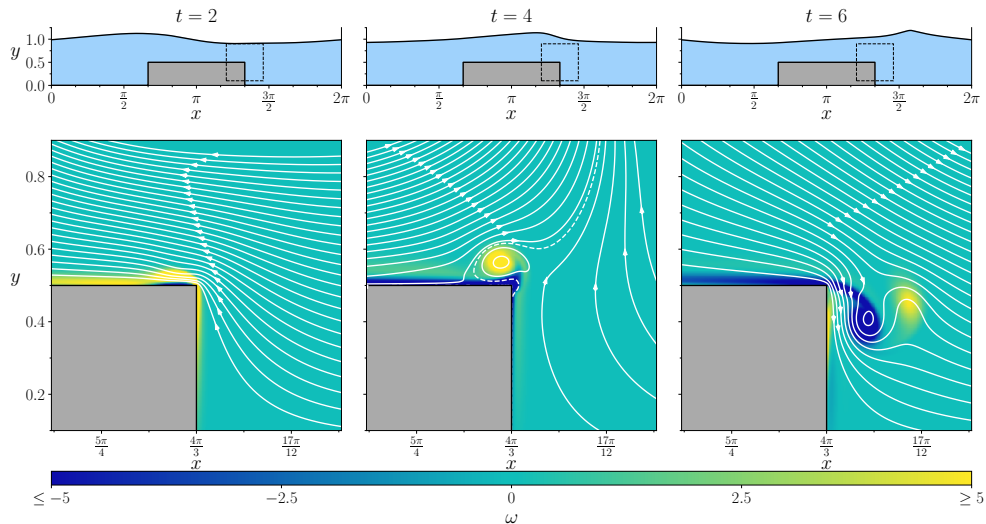
Comparing the interfaces



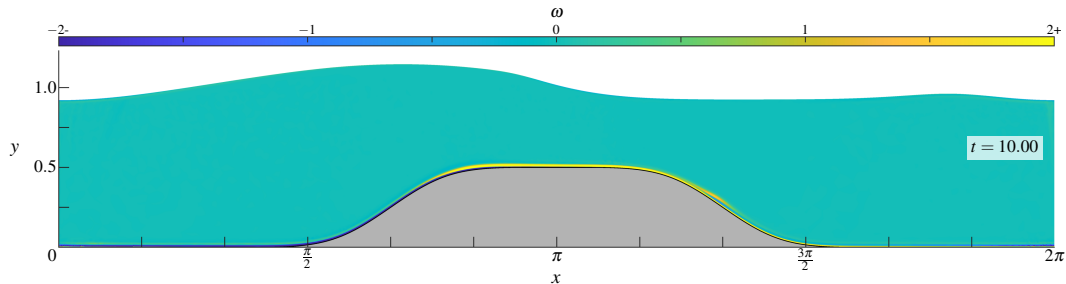
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The flow at $\text{Re} = 10^4$

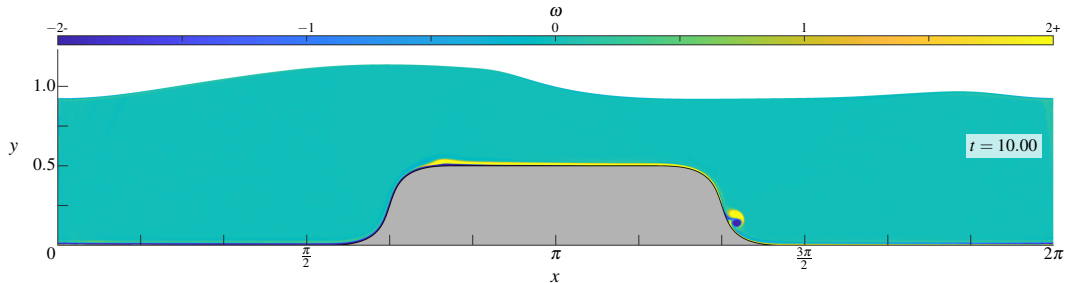


What about a smooth edge?



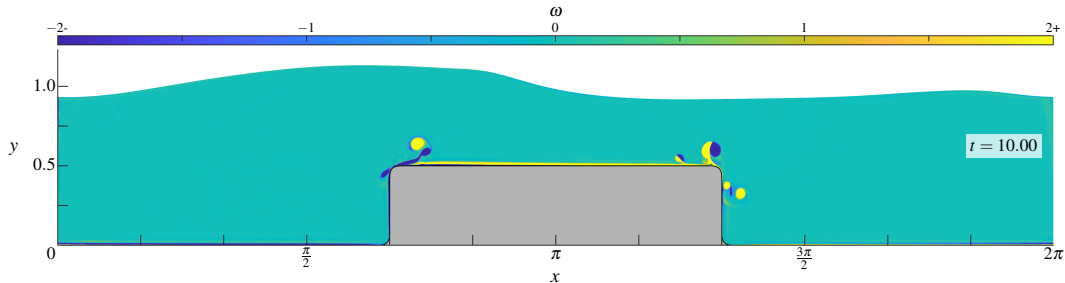
$$r = 1 \quad \text{and} \quad \text{Re} = 10^5$$

What about a smooth edge?



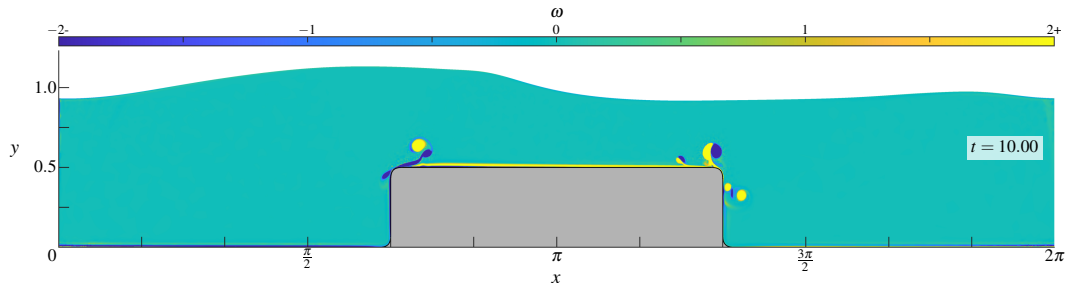
$$r = 0.5 \quad \text{and} \quad \text{Re} = 10^5$$

What about a smooth edge?



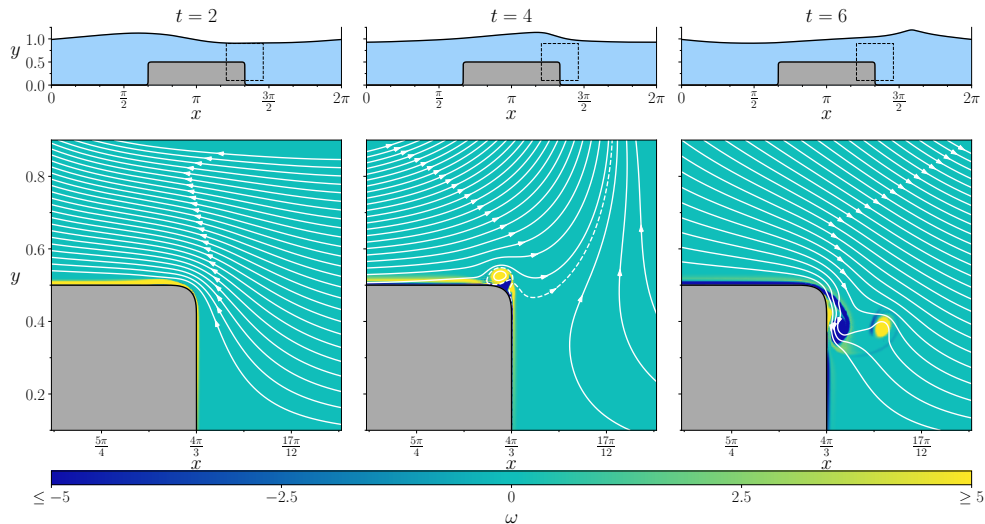
$$r = 0.1 \quad \text{and} \quad \text{Re} = 10^5$$

Animation at $\text{Re} = 10^5$ and $r = 0.1$



If the video does not play, click [here](#).

The flow at $\text{Re} = 10^5$ and $r = 0.1$



Conclusions

free-surface Navier-Stokes



$\text{Re} \rightarrow +\infty$



deep water:  Masmoudi & Rousset (2017)
slip/Navier bottom: ?
no-slip/Dirichlet bottom: ???

free-surface Euler



the flow is initially irrotational

Water Waves

Conclusions

free-surface Navier-Stokes



$Re \rightarrow +\infty$



deep water:  Masmoudi & Rousset (2017)
slip/Navier bottom: **should work!**
no-slip/Dirichlet bottom: ???

free-surface Euler



the flow is initially irrotational

Water Waves

Conclusions

free-surface Navier-Stokes



$\text{Re} \rightarrow +\infty$



deep water:  Masmoudi & Rousset (2017)
slip/Navier bottom: **should work!**
no-slip/Dirichlet bottom: **will not work!**

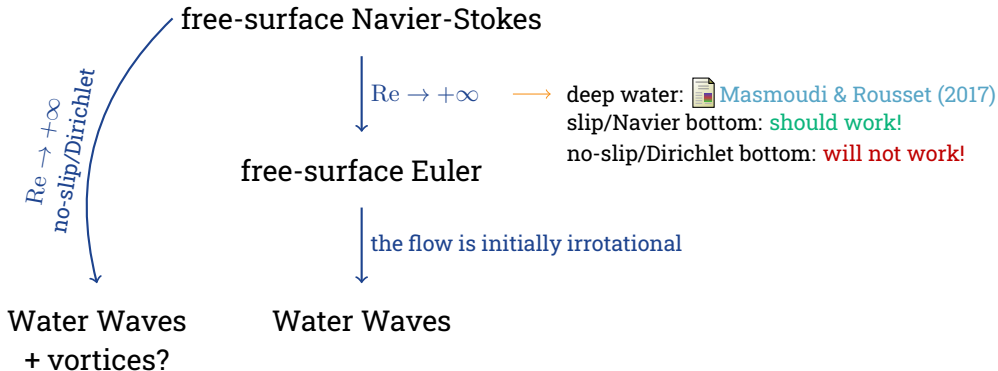
free-surface Euler



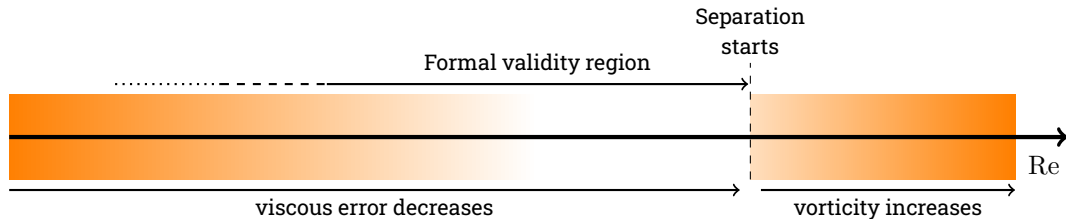
the flow is initially irrotational

Water Waves

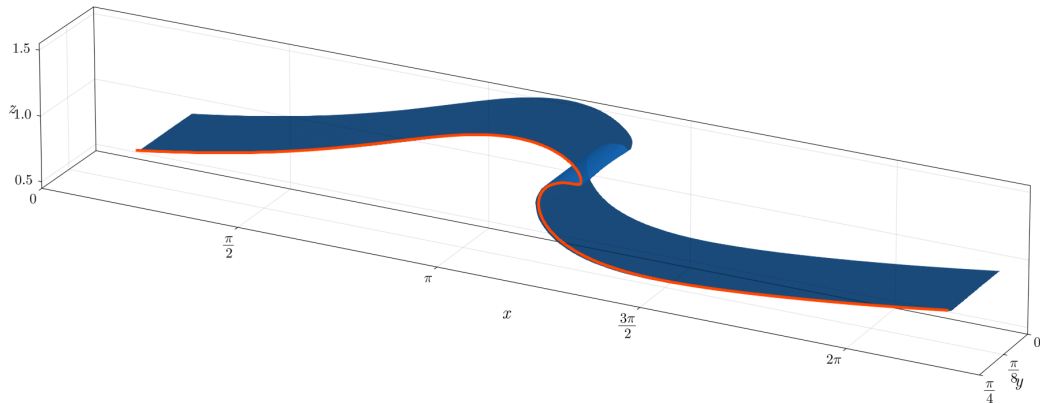
Conclusions



Validity of asymptotic irrotational models



Bonus: 3d simulation!



If the video does not play, click [here](#).

Thank you!

Viscosity and Breaking Waves:  A. R. & E. Dormy (2024) **Numerical study of a viscous breaking water wave and the limit of vanishing viscosity**, J. Fluid Mech. (Rapids) **984**, R5

Irrotationality:  A. R. & E. Dormy (2024) **Irrotationality of Water Waves and Topography**, Submitted. [arXiv:2411.09291](https://arxiv.org/abs/2411.09291) [physics.flu-dyn]

Navier-Stokes around curved obstacles

$$\begin{aligned}\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla p + \frac{1}{\text{Re}} \Delta \mathbf{u} \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

Navier-Stokes around curved obstacles

In the **Frénet-Serre** frame (s, n) , with $h_n = 1 + \kappa(s)n$,

$$\begin{aligned}\partial_t u_s + \frac{1}{h_n} u_s \partial_s u_s + u_n \partial_n u_s + \frac{\kappa}{h_n} u_s u_n &= -\frac{\partial_s p}{h_n} + \frac{1}{\text{Re}} \left[\frac{1}{h_n^2} \partial_{ss} u_s + \partial_{nn} u_s \right. \\ &\quad \left. + \frac{u_n \partial_s \kappa - n \partial_s \kappa \partial_s u_s}{h_n^3} + \frac{\kappa}{h_n} \partial_n u_s + \frac{2\kappa \partial_s u_n - \kappa^2 u_s}{h_n^2} \right] \\ \partial_t u_n + \frac{1}{h_n} u_s \partial_s u_n + u_n \partial_n u_n + \frac{\kappa u_s^2}{h_n} &= -\partial_n p + \frac{1}{\text{Re}} \left[\frac{1}{h_n^2} \partial_{ss} u_n + \partial_{nn} u_n \right. \\ &\quad \left. + \frac{u_s \partial_s \kappa - n \partial_s \kappa \partial_s u_n}{h_n^3} + \frac{\kappa}{h_n} \partial_n u_n - \frac{2\kappa \partial_s u_s + \kappa^2 u_n}{h_n^2} \right] \\ \partial_s u_s + \partial_n (h_n u_n) &= 0\end{aligned}$$

Navier-Stokes around curved obstacles

Prandtl's system,

$$\begin{aligned}\partial_t u_s + u_s \partial_s u_s + u_n \partial_N u_s + \partial_s p - \partial_{NN} u_s &= O(\beta) + O(\text{Re}^{-1}) \\ \partial_N p &= O(\beta) + O(\text{Re}^{-1}) \\ \partial_s u_s + \partial_N u_n &= O(\beta),\end{aligned}$$

with $\delta N = n$ the rescaled normal coordinate and the parameters

$$\text{Re} = \frac{UL}{\nu} \quad \text{and} \quad \beta = \kappa_{\max} \delta = \frac{\kappa_{\max} L}{\sqrt{\text{Re}}}$$